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Solvability of the Boundary Value Problem Coordinated with the Anisotropic Helmholt-Shrodinger Equation

S.A. Hosseini Matikolai

Department of Mathematics, Yerevan State University, Yerevan, Armenia

Abstract: In this paper, solvability of one the anisotropic Helmholt-Shrodinger equation with the boundary conditions of the first and second type is investigated in the upper and the lower half-plan ($y > 0$, $y < 0$). In general, necessary and sufficient conditions for the correctness of the problem in the Sobolev space are presented as well as explicit formulas for a factorization of the Fourier symbol matrix of the one-medium problem.

Key words: Helmholtz-Shrodinger equation · Factorization of matrix-function · Boundary value problem[1]

INTRODUCTION

Investigated a certain class of diffraction problems leading to simultaneous 2 × 2 systems of Wiener-Hopf equations. First the classical Wiener-Hopf technique, represented by Noble [1].

This type of Problems studied by A.J.Sommerfeld for the wave diffraction on the interface of two media [2, 3] and were investigated in the isotropic case [3, 4] and studied the problem of finding a function u in a suitable space with satisfies [3].

Various physical problem in diffraction theory lead us to study modification of the Sommerfeld half-plane governed by two proper elliptic partial differential equation is complementary R^3 half-space $\Omega^{\pm}$ and allow different boundary or transmission condition on two half-lanes, which together form the common boundary of $\Omega^{\pm}$ [3].

In this paper we investigate solvability of the boundary value problem coordinated with the anisotropic Hemholt-Shrodinger equation, in the sobolev spaces. Further we prove that solvability of the boundary value problem is equivalent to the solvability of the some Riemann-Hilbert problem.

Convention: As a rule, upper or lower indices ± are related to the half-spaces $\Omega^{\pm}$ except for some standard notation $R_{\pm}$ and $H^{\pm\frac{1}{2}}$.

Investigate solvability of anisotropic boundary value problem.

Consider the following anisotropic Helmholtz-Schrodinger equation

$$\begin{cases} \Delta u + \left(k_+^2 + 2\beta_+^2 \operatorname{sech}^2(\beta_+ y)\right) u = 0 \quad \text{in } \Omega^+ \\ \Delta u + \left(k_-^2 + 2\beta_-^2 \operatorname{sech}^2(\beta_- y)\right) u = 0 \quad \text{in } \Omega^- \end{cases} \tag{1}$$

Let $\Omega^{\pm} = \{(x, y) \in R^2 : y > 0, y < 0\}$, where Im $(k_{\pm}) > 0$ and let $H^{1/2}(\Omega^{\pm})$ and $H^{-1/2}(\Omega^{\pm})$ are the corresponding sobolev spaces (see [5].

Now we suppose the following boundary conditions

$$\begin{cases} \begin{cases} a_0 u(x,+0) + b_0 u(x,-0) = h_0(x) \\ a_1 \dfrac{\partial u(x,+0)}{\partial y} + b_1 \dfrac{\partial u(x,-0)}{\partial y} = h_1(x) \end{cases} \quad \text{in } R^+ \\ \begin{cases} c_0 u(x,+0) + d_0 u(x,-0) = p_0(x) \\ c_1 \dfrac{\partial u(x,+0)}{\partial y} + d_1 \dfrac{\partial u(x,-0)}{\partial y} = p_1(x) \end{cases} \quad \text{in } R^- \end{cases} \tag{2}$$

[1]2000 Mathematics subject classifications: 47A68, 47A70.

where $h_0 \in H^{1/2}(R^+)$, $p_0 \in H^{1/2}(R^-)$, $p_1 \in H^{1/2}(R^-)$ and a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1, are complex constants. For finding the solution of the boundary value problem (1) in the $L^2(R^2)$, apply Fourier integral transform to the solution $u \in L^2(L^2)$, over the variable x one derives the following system of ordinary differential equations

$$\begin{cases} \dfrac{d^2\hat{u}}{dy^2} + \left(\kappa_+^2(\lambda) + 2\beta_+^2 \operatorname{sech}^2(\beta_+ y)\right)\hat{u} = 0, & \text{for } y > 0 \\ \dfrac{d^2\hat{u}}{dy^2} + \left(\kappa_-^2(\lambda) + 2\beta_-^2 \operatorname{sech}^2(\beta_- y)\right)\hat{u} = 0, & \text{for } y < 0. \end{cases} \tag{3}$$

Then $\hat{u} \in L^2(L^2)$, we denote $\gamma_\pm(\lambda) = \sqrt{\lambda^2 - k_\pm^2} = i\kappa_\pm(\lambda) = \sqrt{k_\pm^2 - \lambda^2}$.

It follows that the general solution of the system of ordinary differential equations (3) in the $L^2(L^2)$-space has the following form:

$$\hat{u}(\lambda, y) = \begin{cases} a(\lambda)\dfrac{i\kappa_+(\lambda) - \beta_+ \tanh(\beta_+ y)}{i\kappa_+(\lambda)} e^{i\kappa_+(\lambda) y}, & \text{for } y > 0 \\ b(\lambda)\dfrac{i\kappa_-(\lambda) + \beta_- \tanh(\beta_- y)}{i\kappa_-(\lambda)} e^{-i\kappa_-(\lambda) y}, & \text{for } y < 0. \end{cases} \tag{4}$$

Let $\chi_\pm(y) = 1/2(1 \pm \operatorname{sgn} y)$ and

$$\begin{cases} \hat{u}_+(\lambda, y) = \dfrac{\chi_+(y)}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} u(x, y) e^{i\lambda x} dx \\ \hat{u}_-(\lambda, y) = \dfrac{\chi_-(y)}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} u(x, y) e^{i\lambda x} dx. \end{cases} \tag{5}$$

Then from eq.(4) it follows that

$$\hat{u}(\lambda, y) = \hat{u}_+(\lambda, y) + \hat{u}_-(\lambda, y) \tag{6}$$

we introduce the following notations:

$$\begin{cases} u_-(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{0} (a_0 u(x, +0) + b_0 u(x, -0) - h_0(x)) e^{i\lambda x} dx \\ w_-(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{0} (a_1 \dfrac{\partial u(x, +0)}{y} + b_1 \dfrac{\partial u(x, -0)}{y} - h_1(x)) e^{i\lambda x} dx, \end{cases} \tag{7}$$

similarly

$$\begin{cases} u_+(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{\infty} (c_0 u(x, +0) + d_0 u(x, -0) - p_0(x)) e^{i\lambda x} dx \\ w_+(\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{\infty} (c_1 \dfrac{\partial u(x, +0)}{y} + d_1 \dfrac{\partial u(x, -0)}{y} - p_1(x)) e^{i\lambda x} dx. \end{cases} \tag{8}$$

So

$$\frac{d\hat{u}(\lambda,y)}{dy}=\begin{cases} a(\lambda)\left[(i\kappa_+(\lambda)-\beta_+\tanh(\beta_+y))-\dfrac{\beta_+^2}{i\kappa_+(\lambda)\cosh^2(\beta_+y)}\right]e^{i\kappa_+(\lambda)y}, & \text{for } y>0 \\ -b(\lambda)\left[(i\kappa_-(\lambda)+\beta_-\tanh(\beta_-y))+\dfrac{\beta_-^2}{i\kappa_-(\lambda)\cosh^2(\beta_-y)}\right]e^{-i\kappa_-(\lambda)y}, & \text{for } y<0. \end{cases} \tag{9}$$

Using boundary conditions (2) and taking into account eqs.(4), (9) one derives

$$\begin{cases} a_0a(\lambda)+b_0b(\lambda)=u_-(\lambda)+\hat{h}_0(\lambda) \\ \dfrac{-a_1[\kappa_+^2(\lambda)+\beta_+^2]a(\lambda)}{i\kappa_+(\lambda)}+\dfrac{b_1[\kappa_-^2(\lambda)+\beta_-^2]b(\lambda)}{i\kappa_-(\lambda)}=w_-(\lambda)+\hat{h}_1(\lambda), \end{cases} \tag{10}$$

Where

$$\hat{h}_0(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}h_0(x)e^{i\lambda x}dx,\ \hat{h}_1(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}h_1(x)e^{i\lambda x}dx$$

Assume that the determinant $\Delta(\lambda)$ of system (10) is not zero, i.e.,

$$\Delta(\lambda)=a_0b_1\frac{\kappa_-^2(\lambda)+\beta_-^2}{i\kappa_-(\lambda)}+a_1b_0\frac{\kappa_+^2(\lambda)+\beta_+^2}{i\kappa_+(\lambda)}=a_0b_1\frac{\gamma_-^2(\lambda)-\beta_-^2}{\gamma_-(\lambda)}+a_1b_0\frac{\gamma_+^2(\lambda)-\beta_+^2}{\gamma_+(\lambda)}\neq 0 \tag{11}$$

In view of eq (10):

$$\begin{cases} a(\lambda)=\dfrac{1}{\Delta(\lambda)}\left\{b_1\dfrac{\kappa_-^2(\lambda)+\beta_-^2}{i\kappa_-(\lambda)}(u_-(\lambda)+\hat{h}_0(\lambda))-b_0\left(w_-(\lambda)+\hat{h}_1(\lambda)\right)\right\} \\ b(\lambda)=\dfrac{1}{\Delta(\lambda)}\left\{a_1\dfrac{\kappa_+^2(\lambda)+\beta_+^2}{i\kappa_+(\lambda)}(u_-(\lambda)+\hat{h}_0(\lambda))+a_0\left(w_-(\lambda)+\hat{h}_1(\lambda)\right)\right\}, \end{cases} \tag{12}$$

then, taking into account that

$$\begin{cases} u_+(\lambda)=c_0a(\lambda)+d_0b(\lambda)-\hat{p}_0(\lambda) \\ w_+(\lambda)=\dfrac{-c_1[\kappa_+^2(\lambda)+\beta_+^2]a(\lambda)}{i\kappa_+(\lambda)}+\dfrac{d_1[\kappa_-^2(\lambda)+\beta_-^2]b(\lambda)}{i\kappa_-(\lambda)}-\hat{p}_1(\lambda), \end{cases} \tag{13}$$

Where

$$\hat{p}_0(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}p_0(x)e^{i\lambda x}dx,\ \hat{p}_1(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}p_1(x)e^{i\lambda x}dx$$

which derives the following boundary problem of Riman-Hilbert with respect to the

$$\vec{u}_+(\lambda)=\begin{pmatrix}u_+(\lambda)\\ w_+(\lambda)\end{pmatrix},\vec{u}_-(\lambda)=\begin{pmatrix}u_-(\lambda)\\ w_-(\lambda)\end{pmatrix} \tag{14}$$

$\vec{u}_+(\lambda)$ *and* $\vec{u}_-(\lambda)$ are analytical functions in the upper and lower semiplanes respectively in the vector notations this problem takes the following form:

$$\vec{u}_+(\lambda) = L(\lambda)\vec{u}_-(\lambda) + \vec{m}(\lambda) \tag{15}$$

Where the matrix function $L(x)$ is:

$$L(\lambda) = \frac{1}{\Delta(\lambda)}\begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) \end{pmatrix} \tag{16}$$

With

$$A_{11}(\lambda) = a_1 d_0 \frac{\kappa_+^2(\lambda) + \beta_+^2}{i\kappa_+(\lambda)} + b_1 c_0 \frac{\kappa_-^2(\lambda) + \beta_-^2}{i\kappa_-(\lambda)} = a_1 d_0 \frac{\gamma_+^2(\lambda) - \beta_+^2}{\gamma_+(\lambda)} + b_1 c_0 \frac{\gamma_-^2(\lambda) - \beta_-^2}{\gamma_-(\lambda)}$$

$$A_{12}(\lambda) = a_0 d_0 - b_0 c_0$$

$$A_{21}(\lambda) = (a_1 d_1 - b_1 c_1)\frac{\kappa_+^2(\lambda) + \beta_+^2}{i\kappa_+(\lambda)} \times \frac{\kappa_-^2(\lambda) + \beta_-^2}{i\kappa_-(\lambda)} = (a_1 d_1 - b_1 c_1)\frac{\gamma_+^2(\lambda) - \beta_+^2}{\gamma_+(\lambda)} \times \frac{\gamma_-^2(\lambda) - \beta_-^2}{\gamma_-(\lambda)}$$

$$A_{22}(\lambda) = b_0 c_1 \frac{\kappa_+^2(\lambda) + \beta_+^2}{i\kappa_+(\lambda)} + a_0 d_1 \frac{\kappa_-^2(\lambda) + \beta_-^2}{i\kappa_-(\lambda)} = b_0 c_1 \frac{\gamma_+^2(\lambda) - \beta_+^2}{\gamma_+(\lambda)} + a_0 d_1 \frac{\gamma_-^2(\lambda) - \beta_-^2}{\gamma_-(\lambda)} \; .$$

The coordinates of the vector-function $\vec{m}(\lambda)$

$$\vec{m}(\lambda) = \begin{pmatrix} m_1(\lambda) \\ m_2(\lambda) \end{pmatrix} \tag{17}$$

have the following form

$$m_1(\lambda) = \frac{\hat{h}_0(\lambda)}{\Delta(\lambda)}\left\{ a_1 d_0 \frac{\kappa_+^2(\lambda) + \beta_+^2}{i\kappa_+(\lambda)} + b_1 c_0 \frac{\kappa_-^2(\lambda) + \beta_-^2}{i\kappa_-(\lambda)} \right\} + \frac{a_0 d_0 - b_0 c_0}{\Delta(\lambda)} \hat{h}_1(\lambda) - \hat{p}_0(\lambda)$$

$$m_2(\lambda) = \frac{\hat{h}_1(\lambda)}{\Delta(\lambda)}\left\{ b_0 c_1 \frac{\kappa_+^2(\lambda) + \beta_+^2}{i\kappa_+(\lambda)} + a_0 d_1 \frac{\kappa_-^2(\lambda) + \beta_-^2}{i\kappa_-(\lambda)} \right\} +$$

$$\frac{a_1 d_1 - b_1 c_1}{\Delta(\lambda)} \hat{h}_0(\lambda) \frac{\kappa_+^2(\lambda) + \beta_+^2}{i\kappa_+(\lambda)} \times \frac{\kappa_-^2(\lambda) + \beta_-^2}{i\kappa_-(\lambda)} - \hat{p}_1(\lambda)$$

The case of $\beta_- = \beta_+ = 0$ was studied in the papers [6, 7].

Theorem: *If the function $u \in L^2(L^2)$ is the solution of the boundary problem (1), then the pair of vector-functions $\vec{u}_+(\lambda)$ and $\vec{u}_-(\lambda)$ is a solution of the boundary problem of Riman-Hilbert (15). Vice-versa, if one applies inverse Fourier transform to the function $\hat{u}(\lambda, y) = \hat{u}_+(\lambda, y) + \hat{u}_-(\lambda, y)$, which is associated with vector functions $\vec{u}_+(\lambda)$ and $\vec{u}_-(\lambda)$ by the relations (7), (8), then solution of the boundary problem (1), will be derived.*

Theory of the Riman-Hilbert problem solvability is demonstrated in the monographs [6, 8]. The essence of this problem solution is linked to the problem of the $L(\lambda)$ matrix-function factorization, i.e. to the problem of its presentation as

$$L(\lambda) = L_+(\lambda)\Lambda(\lambda)L_-(\lambda) \tag{18}$$

Where the matrix-functions $L_+(\lambda)$, $L_+^{-1}(\lambda)$ are analytic in the upper semiplane and $L_-(\lambda)$, $L_-^{-1}(\lambda)$ are analytic in the lower semiplane. The diagonal matrix-function can be presented as

$$\Lambda(\lambda) = \begin{pmatrix} \left(\frac{\lambda - i}{\lambda + i}\right)^{\alpha_1} & 0 \\ 0 & \left(\frac{\lambda - i}{\lambda + i}\right)^{\alpha_2} \end{pmatrix} \tag{19}$$

Where α_1 and α_2 are partial indices of the given diagonal matrix-function, which allow one to write the Fredholm characteristics of the initial boundary problem.

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