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Tangent Developable of Biharmonic Curve in the Special Three-Dimensional ϕ–Ricci Symmetric Para-Sasakian Manifold P

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Abstract: In this paper, we study in particular developable surfaces, a special type of ruled surface in the special three-dimensional ϕ– Ricci symmetric para-Sasakian manifold P. We characterize biharmonic curves in terms of their curvature and torsion and we prove that all of biharmonic curves are helices in the special three-dimensional ϕ–Ricci symmetric para-Sasakian manifold P. Moreover, we find out explicit parametric equations of tangent developable of biharmonic curve in the special three-dimensional ϕ–Ricci symmetric para-Sasakian manifold P.

Mathematics Subject Classifications: 53C41, 53A10.
Key words: Biharmonic curve · Para-Sasakian manifold · Curvature · Torsion · Developable surface

INTRODUCTION

The geometry of curves and surfaces has long captivated the interests of mathematicians, from the ancient Greeks through to the era of Isaac Newton (1643-1727) and the invention of the calculus. During the 18th century, through the use of differential calculus, the study of geometry evolved into differential geometry and it is in this context that the reader is presented with the topic of this report, ruled surfaces.

Ruled surfaces and especially developable surfaces are well-known and widely used in computer aided design and manufacture. Since these fields apply B-spline or NURBS surfaces as de facto standard description methods, it is highly desired to use these methods to construct ruled and developable surfaces from any type of data. These data can be scattered points or given lines or a set of tangent planes, as well. Since these special surfaces possess a wide range of applications, e.g. from ship hulls to sheet metal forming processes, one can find several algorithms solving this problem, [1,2].

It is well known that developable surfaces play an important role in design in several branches of industry, such as naval and textile. Even architectural structures have been designed using developable surfaces. In these industries surfaces are designed which mimic properties of the materials that are used in production, which are intended to be deformed from plane sheets of metal or cloth just by folding, cutting or rolling, but not stretching. This sort of industrial procedures are less expensive or do not alter the properties of the material an therefore developable surfaces are favoured.

The aim of this paper is to study tangent developable of biharmonic curve in the special three-dimensional ϕ–Ricci symmetric para-Sasakian manifold P.

Let φ : $(M,g)\to(N,h)$ be a smooth map between Riemannian manifolds with M compact. Then φ is called biharmonic if it is an extremal of the functional

$$E_2(\varphi)=\frac{1}{2}\int_M\left|\tau(\varphi)\right|^2 v^M,$$

Where $\tau(\varphi)$ denotes the tension field of the map φ and v^M is the volume form on M, [3-11]. Clearly every harmonic map is biharmonic (see [8] for a background on harmonic maps). If we set

$$E_1(\varphi)=\frac{1}{2}\int_M\left|d\varphi\right|^2 v^M,$$

to be the energy of φ, then we recall the first variation formula

$$\frac{\partial}{\partial s}E_1(\varphi_s)|_{s=0}=-\int_M\left\langle\tau(\varphi),v\right\rangle v^M,$$

Where:

$$v=\frac{\partial\varphi}{\partial s}|_{s=0}\in\Gamma\left(\varphi^{-1}TN\right)$$

is an arbitrary variation of $\varphi=\varphi_0$. If v is taken to be in the direction of $\tau(\varphi)$, then

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$$\frac{\partial}{\partial s}E_1(\varphi_s)|_{s=0} = -\int_M |\tau(\varphi)|^2 v^M = -E_2(\varphi).$$

Now take an arbitrary variation of $E_2(\varphi)$ in the direction $w = \frac{\partial\varphi}{\partial t}|_{t=0}$, we have

$$\frac{\partial}{\partial t}E_2(\varphi_t)|_{t=0} = -\frac{\partial^2}{\partial s \partial t}E_1(\varphi_{s,t})|_{s,t=0}$$
$$= -\int_M \langle J_\varphi(\tau(\varphi)), v\rangle v^M,$$

Where J_φ is the Jacobi operator corresponding to the second variation of $E_1(\varphi)$. The Euler--Lagrange equations for a biharmonic map are therefore given by the negative of the Jacobi operator acting on the tension field:

$$\tau_2(\varphi) \equiv -\mathrm{Tr}_g\left(\nabla^\varphi\right)^2 \tau(\varphi) - \mathrm{Tr}_g R^N\left(\tau(\varphi), d\varphi\right)d\varphi = 0. \tag{1.1}$$

Here, our convention for the curvature is

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and

$$\left(\nabla^\varphi\right)^2_{X,Y} v = \nabla^\varphi_X\left(\nabla^\varphi_Y v\right) - \nabla^\varphi_{\nabla^M_X Y} v$$

With ∇^φ representing the connection in the pull-back bundle $\varphi^{-1}(TN)$ and ∇^M the Levi-Civita connection on M More generally, in the case when M is no longer compact, we call a smooth map φ biharmonic if it satisfies (1.1), [12-15].

In this paper, we study in particular developable surfaces, a special type of ruled surface in the special three-dimensional φ – Ricci symmetric para-Sasakian manifold P. We characterize biharmonic curves in terms of their curvature and torsion and we prove that all of biharmonic curves are helices in the special three-dimensional φ – Ricci symmetric para-Sasakian manifold P. Moreover, we find out explicit parametric equations of tangent developable of biharmonic curve in the special three-dimensional φ – Ricci symmetric para-Sasakian manifold P.

Preliminaries: An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi = 0, \eta(\xi) = 1, g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

For any vector fields X, Y on M.

In addition, if (ϕ, ξ, η, g), satisfy the equations

$$d\eta = 0, \nabla_X\xi = \phi X, \tag{2.4}$$

$$(\nabla_X\phi)Y = -g(X,Y)\xi - \eta(Y)$$
$$X + 2\eta(X)\eta(Y)\xi, X,Y \in \chi(M), \tag{2.5}$$

Then M is called a para-Sasakian manifold or, briefly a P- Sasakian manifold. In particular, a P- Sasakian manifold M is called a special para-Sasakian manifold or briefly a SP- Sasakian manifold if M admits a 1-form η satisfying

$$(\nabla_X\eta)Y = -g(X,Y) + \eta(X)\eta(Y). \tag{2.6}$$

It is known [16-17] that in a P-Sasakian manifold the following relations hold:

$$S(X,\xi) = -(n-1)\eta(X),$$
$$Q\xi = -(n-1)\xi,$$
$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$
$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi,$$
$$R(\xi,X)\xi = X - \eta(X)\xi,$$
$$\eta(R(X,Y)Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z),$$
$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y),$$

For any vector fields X, Y, Z on M.

A para-Sasakian manifold is said to be Einstein if the Ricci tensor λ is of the form

$$S(X,Y) = \lambda g(X,Y)$$

Where λ is a constant.

Special Three-dimensional ϕ–Ricci Symmetric Para-sasakian Manifold P

Definition 3.1: A para-Sasakian manifold M is said to be locally ϕ-symmetric if

$$\phi^2\left((\nabla_W R)(X,Y)Z\right) = 0,$$

For all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by Takahashi [16], for a Sasakian manifold.

Definition 3.2: A para-Sasakian manifold M is said to be ϕ-symmetric if

$$\phi^2\left((\nabla_W R)(X,Y)Z\right)=0,$$

For all vector fields X, Y, Z, W on M

Definition 3.3: A para-Sasakian manifold M is said to be ϕ-Ricci symmetric if the Ricci operator satisfies

$$\phi^2\left((\nabla_X Q)(Y)\right)=0,$$

For all vector fields X and Y on M and $S(X,Y) = g(QX,Y)$.

If X,Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

We consider the three-dimensional manifold

$$P = \{(x^1,x^2,x^3,) \in R^3 : (x^1,x^2,x^3,) \neq (0,0,0)\}$$

Where $(x^1,x^2,x^3,)$ are the standard coordinates in R^3. We choose the vector fields

$$\mathbf{e}_1 = e^{x^1}\frac{\partial}{\partial x^2},\ \mathbf{e}_2 = e^{x^1}\left(\frac{\partial}{\partial x^2}-\frac{\partial}{\partial x^3}\right),\ \mathbf{e}_3 = -\frac{\partial}{\partial x^1} \tag{3.1}$$

are linearly independent at each point of P.
Let g be the Riemannian metric defined by

$$\begin{aligned} &g(e_1, e_1) = g(e_2\ e_2 = g(e_3\ , e_3) =1, \\ &g(e_1, e_2) = g(e_2\ e_3 = g(e_1\ , e_3) =0. \end{aligned} \tag{3.2}$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z,\mathbf{e}_3) \text{for any } Z \in \chi(P)$$

Let be the (1,1) tensor field defined by

$$\phi(e_1) = e_2, \phi(e_2) = e_1, \phi(e_3) = 0. \tag{3.3}$$

Then using the linearity of and g we have

$$\eta(e_3) = 1 \tag{3.4}$$

$$\phi^2(Z) = Z - \eta(Z)e_3 \tag{3.5}$$

$$g(\phi Z,\phi W) = g(Z,W) - \eta(Z)\eta(W), \tag{3.6}$$

For any $Z, W \in \chi(P)$.. Thus for $e_3 = \xi$, $(\phi,\xi\,\eta,g)$ defines an almost para-contact metric structure on P.

Let ∇ be the Levi-Civita connection with respect to g. Then, we have

$$e_1,e_2]= = 0,\ [e_1,e_3] = e_1,\ [e_2,e_3] = e_2$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y,Z) &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y) \\ &-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]), \end{aligned}$$

Which is known as Koszul's formula.

Taking $e_3 = \xi$ and using the Koszul's formula, we obtain

$$\begin{aligned} &\nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\mathbf{e}_3, \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, \quad \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\ &\nabla_{\mathbf{e}_2}\mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3, \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\ &\nabla_{\mathbf{e}_3}\mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{aligned} \tag{3.7}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,\ R_{ijkl} = R(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_l),$$

Where the indices i, j, k and l take the values 1,2 and 3.

$$R_{122} = -e_1.,\ R_{133} = -e_1.,\ R_{233} = -e_2.,$$

And

$$R_{1212} = R_{1313} = R_{2323} = 1 \tag{3.8}$$

Biharmonic Curves in the Special Three-dimensional ϕ–Ricci Symmetric Para-sasakian Manifold P:

Biharmonic equation for the curve γ reduces to

$$\nabla_{\mathbf{T}}^3\mathbf{T} - R(\mathbf{T},\nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0, \tag{4.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (4.1).

Let us consider biharmonicity of curves in the special three-dimensional ϕ–Ricci symmetric para-Sasakian manifold P. Let {T,N,P} be the Frenet frame field along γ.

Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\begin{aligned} &\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}, \\ &\nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}, \\ &\nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N}, \end{aligned} \tag{4.2}$$

Where k is the curvature of γ and τ its torsion and

$$\begin{aligned} &g(\mathbf{T},\mathbf{T}) = 1, g(\mathbf{N},\mathbf{N}) = 1, g(\mathbf{B},\mathbf{B}) = 1, \\ &g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{e_1,e_2,e_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{4.3}$$

Theorem 4.1: $\gamma : I \to$ P is a biharmonic curve if and only if

$$\begin{aligned} &k = \text{constant} \neq 0, \\ &k^2 + \tau^2 = 1, \\ &\tau^1 = 0. \end{aligned} \tag{4.4}$$

Proof: Using (4.1) and Frenet formulas (4.2), we have (4.4).

Theorem 4.2: All of biharmonic curves in the special three-dimensional ϕ–Ricci symmetric para-Sasakian manifold P are helices.

Tangent Developable of Biharmonic Curve in the Special Three-dimensional ϕ–Ricci Symmetric Para-sasakian Manifold P: Ruled surfaces are swept out by the motion of a straight line in space. More formally, the image of the map $\hat{\text{O}}_{(\gamma,\delta)} : I\times R \to P$ defined by

$$\hat{O}_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s),\ s\in I, u\in R$$

is called a ruled surface in P where $\gamma : I \to$P, $\delta : I \to$ P\{0} are smooth mappings and I is an open interval or a unit circle S^1.

We call γ the base curve and δ the director curve. The straight lines $u \to \gamma(s) + u\delta(s)$ are called rulings.

Note that we allow our ruled surfaces to possess singular points, that is points at which the partial derivatives of $\hat{\text{O}}_{(\gamma,\delta)}$ are linearly indepedent, i.e. which satisfy

$$\begin{aligned} &\hat{\text{O}}_s(s,u)\times\hat{\text{O}}_u(s,u) = 0 \\ &\Leftrightarrow \left(\gamma'(s) + u\delta'(s)\right)\times\delta(s) = 0 \\ &\Leftrightarrow \gamma'(s)\times\delta(s) + u\delta'(s)\times\delta(s) = 0. \end{aligned}$$

We now consider a special type of ruled surface, which has been studied for over a century, the developable surface. Informally, these are surfaces which can be attened onto a plane without distortion, so are a transformation (e.g. folding or bending) of a plane in P. It is this fundamental property which has long ensured their useful application in engineering and manufacturing. More recently, their use has spread to the computer sciences, in computer-aided design; their isometric properties make them ideal primitives for texture mapping.

Definition 5.1: *A smooth surface* $\hat{\text{O}}_{(\gamma,\delta)}$ *is called a developable surface if its Gaussian curvature K vanishes everywhere on the surface.*

Proposition 5.2: *A ruled surface is a developable surface [4] if:*

$$\left(\gamma'(s)\times\delta(s)\right)\times\delta'(s) = 0. \tag{5.1}$$

We can give a geometric interpretation of Proposition 5.2 by computing the Gaussian curvature at a regular point. Since

$$\hat{\text{O}}_{tt} = \gamma'' + u\delta'', \hat{\text{O}}_{tu}\delta' = \delta', \hat{\text{O}}_{uu} = 0, \tag{5.2}$$

computations of the coeffecients of the second fundamental form give:

$$\begin{aligned} N &= 0, \\ M &= \mathbf{n}.\hat{\text{O}}_{tu} \\ &= \frac{(\hat{\text{o}}_t\times\hat{\text{o}}_u).\hat{\text{o}}_{tu}}{|\hat{\text{o}}_t\times\hat{\text{o}}_u|} \\ &= \frac{\left(\gamma'\times\delta\right).\delta'}{|\hat{\text{o}}_t\times\hat{\text{o}}_u|^2} \end{aligned} \tag{5.3}$$

Thefore, at regular points, the Gaussian curvature of a developable surface is zero, which is consistent with The tangent developable of γ is a ruled surface

$$\hat{\text{O}}_{\left(\gamma,\gamma'\right)}(s,u) = \gamma(s) + u\gamma'(s).. \tag{5.4}$$

The tangent developable is the envelope of the family of osculating planes along γ, where the osculating plane at $\gamma(s)$ is the plane generated by the tangent vector $\gamma'(s)$ and the principal normal N(s).

Theorem 5.3: *Let* $\gamma : I \to$P *be a unit speed biharmonic curve in the special three-dimensional* ϕ*–Ricci symmetric para-Sasakian manifold* P. *Then, the parametric equations for tangent developable* of γ *are*

$$x^1(s,u) = -s\cos\varphi - u\cos\varphi + C_1,$$

$$x^2(s,u) = C_2 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + C_1}\Big(\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi} + \cos\varphi\right]\cos\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right]$$
$$+\left[-\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi} + \cos\varphi\right]\sin\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right]\Big) + u\sin\varphi e^{-s\cos\varphi}\left(\sin\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right] + \cos\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right]\right),$$

$$x^3(s,u) = C_3 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + C_1}\Big(-\cos\varphi\cos\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right] + \left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right]\sin\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right]\Big)$$
$$+u\sin\varphi e^{-s\cos\varphi + C_1}\sin\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right], \tag{5.5}$$

Where C, C_1, C_2, C_3 are constants of integration

Proof: Suppose that $\gamma : I \to P$ be a unit speed non-geodesic curve in the special three-dimensional ϕ–Ricci symmetric para-Sasakian manifold P. Since γ is biharmonic, γ is a helix. So, without loss of generality, we take the axis of γ is parallel to the vector e_3. Then,

$$g(T,e_3) = T_3 = \cos\varphi, \tag{5.6}$$

Where φ is constant angle.
The tangent vector can be written in the following form

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3. \tag{5.7}$$

On the other hand the tangent vector T is a unit vector, so the following condition is satisfied

$$T_1^2 + T_2^2 = 1 - \cos^2\varphi. \tag{5.8}$$

Noting that $\cos^2\varphi + \sin^2\varphi = 1$, we have

$$T_1^2 + T_2^2 = \sin^2\varphi. \tag{5.9}$$

The general solution of (5.9) can be written in the following form

$$T_1 = \sin\varphi\cos\mu, \tag{5.10}$$
$$T_2 = \sin\varphi\sin\mu,$$

Where μ is an arbitrary function of s

So, substituting the components T_1 T_2 and T_3 in the equation (5.7), we have the following equation

$$\mathbf{T} = \sin\varphi\cos\mu\mathbf{e}_1 + \sin\varphi\sin\mu\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{5.11}$$

Since $|\nabla_{\mathbf{T}}\mathbf{T}| = \kappa$, we obtain

$$\mu = \frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C, \tag{5.12}$$

Where $C \in \mathbb{R}$.
Thus (5.11) and (5.12), imply

$$\mathbf{T} = \sin\varphi\cos\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right]\mathbf{e}_1 + \sin\varphi\sin\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{5.13}$$

Using (3.1) in (5.13), we obtain

$$\mathbf{T} = \Big(-\cos\varphi, \sin\varphi e^{x^1}\left(\sin\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right] + \cos\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right]\right), \tag{5.14}$$
$$\sin\varphi e^{x^1}\sin\left[\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}s + C\right]\Big).$$

From third component of T, we have

$$\frac{dx^1}{ds} = -\cos\varphi.$$
$$x^1(s) = -s\cos\varphi + C_1. \tag{5.15}$$

By direct calculations we have

$$\frac{dx^2}{ds} = \sin\varphi e^{-s\cos\varphi + C_1}\left(\sin\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C\right] + \cos\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C\right]\right),$$

$$\frac{dx^3}{ds} = C_1\sin\varphi e^{-s\cos\varphi + C_1}\cos\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C\right].$$

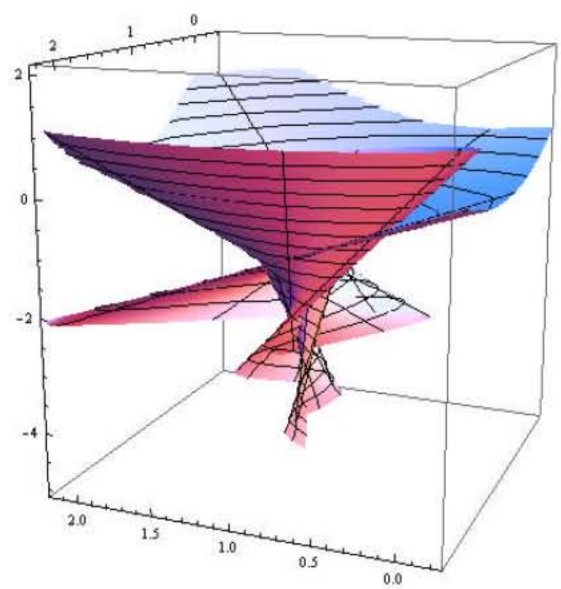
Moreover, above equations, imply

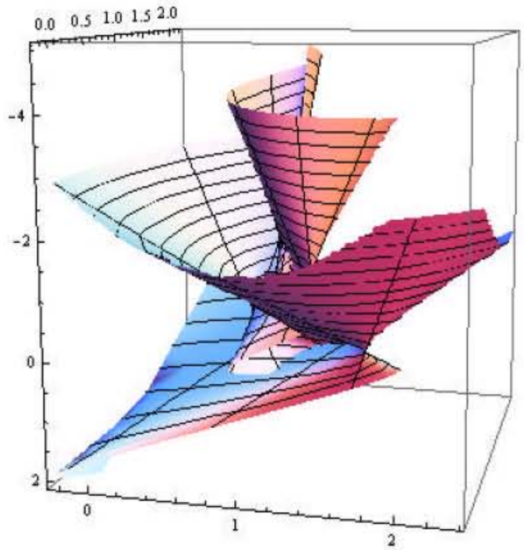
$$x^2(s) = C_2 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi}e^{-s\cos\varphi + C_1}\Bigg(\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi} + \cos\varphi\right]\cos\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C\right] \tag{5.16}$$
$$+\left[-\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi} + \cos\varphi\right]\sin\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C\right]\Bigg),$$

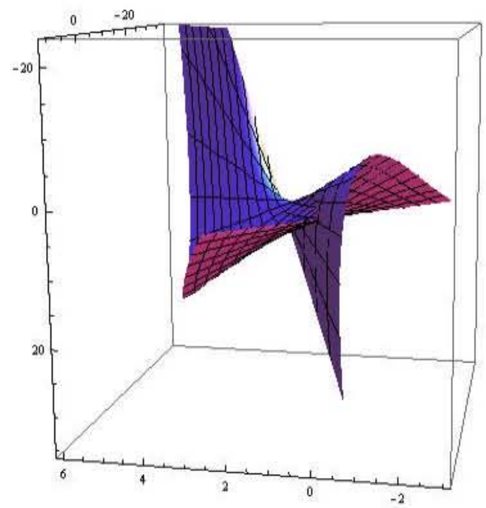
$$x^3(s) = C_3 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi}e^{-s\cos\varphi + C_1}\Bigg(-\cos\varphi\cos\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C\right] \tag{5.17}$$
$$+\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C\right]\sin\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C\right]\Bigg).$$

So, substituting (5.14), (5.15), (5.16) and (5.17) in (5.4), we have (5.5).

If we use Mathematica in Theorem 5.3 for different constant, yields







Corollary 5.4: *Let $\gamma : I \to P$ be a unit speed non-geodesic curve. Then, the parametric equations of γ in terms of tangent developable of γ are*

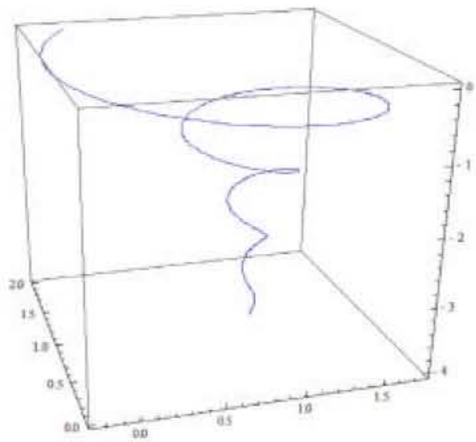
$$x^1(s) = -s\cos\varphi + C_1,$$

$$x^2(s) = C_2 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + C_1} \left(\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi} + \cos\varphi \right] \cos\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi} s + C \right] + \left[-\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi} + \cos\varphi \right] \sin\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi} s + C \right] \right),$$

$$x^3(s) = C_3 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + C_1} \left(-\cos\varphi \cos\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi} s + C \right] + \left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi} s + C \right] \sin\left[\frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi} s + C \right] \right),$$

Where C, C_1 C_2 C_3 are constants of integration

We can use Mathematica, yields



$\cos\varphi = \sin\varphi = \frac{\sqrt{2}}{2}, C = C_1 = C_2 = C_3 = \kappa = 1.$

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