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Modified Decomposition Method for Tenth and Ninth-Order BVPS

[1]Syed Tauseef Mohyud-Din and [2]Ahmet Yildirim

[1]Department of Mathematics, HITEC University Taxila Cantt, Pakistan
[2]Department of Mathematics, Ege University, Science Faculty, 35100 Bornova, Izmir, Turkey

Abstract: In this paper, we apply a relatively new technique which is called the modified decomposition method (MDM) to solve ninth and tenth-order boundary value problems. The suggested algorithm is quite efficient and is practically well suited for use in these problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. Several examples are given to verify the reliability and efficiency of the method. The fact that the proposed MDM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method.

Key words: Modified decomposition method · Higher-order boundary value problems · Error estimates

INTRODUCTION

The (BVPS) boundary value problems of tenth and ninth-order [1-17] arise in the study of astrophysics, hydrodynamic and hydro magnetic stability. Moreover, when a layer of fluid is heated from below and is subject to the action of rotation, instability may set in as ordinary convection which may be modeled by a tenth-order boundary value problem, see [4-6, 10-13] and the references therein. Keeping in view the physical importance of such problems, there is still a dire need to investigate them with an appropriate mathematical algorithm. Recently, Geijji and Jafari [6] introduced a very reliable and efficient technique which is called the modified decomposition method (MDM) and has been applied [6, 18-21] to a wide class of diversified linear and nonlinear problems of physical nature. The basic motivation of this paper is the implementation and extension of this technique (MDM) to find solutions of ninth and tenth-order boundary value problems (BVPS). It is observed that the proposed MDM [6, 18-22] is extremely useful, very simple and highly accurate. It is worth mentioning that the suggested method (MDM) is applied without any discretization, restrictive assumption or transformation and is free from round off errors. The selection of initial value is done very carefully because the approximants are heavily dependant upon the initial value. Unlike the method of separation of variables that require initial and boundary conditions, the method provides an analytical solution by using the initial conditions only. The proposed method work efficiently and the results are very encouraging and reliable. The fact that the proposed MDM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method. Numerical results clearly reveal the complete reliability of the proposed modified decomposition method (MDM).

Modified Decomposition Method (MDM): Consider the following general functional equations:

$$f(x) = 0, \tag{1}$$

To convey the idea of the modified decomposition method [6, 18-22], we rewrite the above equation as:

$$y = N(y) + c, \tag{2}$$

Where N is a nonlinear operator from a banach space $B \to B$ and f is a known function. We are looking for a solution of equation (1) having the series form:

$$y = \sum_{i=0}^{\infty} y_i. \tag{3}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \tag{4}$$

From equations (3) and (4), equation (2) is equivalent to

Correspoding Autor: Dr. Syed Tauseef Mohyud-Din, Department of Mathematics, HITEC University Taxila Cantt, Pakistan.
E-mail: syedtauseefs@hotmail.com.

$$\sum_{i=0}^{\infty} y_i = c + N(y_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \quad (5)$$

We define the following recurrence relation:

$$\begin{cases} y_0 = c, \\ y_1 = N(y_0), \\ y_{m+1} = N(y_0 + ... + y_m) - N(y_0 + ... + y_{m-1}), \ m = 1,2,3,..., \end{cases} \quad (6)$$

then

$$(y1 + ...+ ym+1) = N(y0 +...+ ym), m = 1,2,3,...,$$

and

$$y = f + \sum_{i=1}^{\infty} y_i,$$

if N is a contraction, i.e. $\|N(x) - N(y)\| \le \| x - y \|$, $0 < K < 1$, then

$$\| y_{m+1} \| = \| N(y_0 + ... + y_m) - N(y_0 + ... + y_{m-1}) \| \le K \| y_m \| \le K^m \| y_0 \|,$$
$$m = 0,1,2,3,...,$$

and the series $\sum_{i=1}^{\infty} = y_i$ absolutely and uniformly converges to a solution of equation (1) [6, 18-21], which is unique, in view of the Banach fixed-point theorem.

Numerical Applications: In this section, we apply the modified decomposition method (MDM) to solve BVPS of tenth and ninth-order. The selection of initial value is done carefully because the approximants are heavily dependant upon initial value.

Example 3.1: Consider the following nonlinear boundary value problem of tenth-order

$$y^{(x)}(x) = e^{-x} y^2(x), \quad 0 < x < 1,$$

with boundary condition

$$y(0) = 1, \ \ y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = 1,$$
$$y(0) = e, \ \ y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) = e,$$

The exact solution of the problem is

$$y(x) = e^x.$$

Applying the modified decomposition method (MDM), we get

$$y_{n+1}(x) = y_n(x) + \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left(e^{-x} y_n{}^2(x)\right) dx\,dx\,dx\,dx\,dx\,dx\,dx\,dx\,dx\,dx.$$

Consequently, following approximants are obtained

$$y_0(x) = c,$$
$$y_0(x) = c,$$
$$y_1(x) = N\, y_0(x),$$

$$y_1(x) = Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}x^{11} + \frac{1}{12!}x^{12} + \cdots,$$

$$y_2(x) = N\left(y_0(x) + y_1(x)\right) - N\, y_0(x),$$

$$y_2(x) = \frac{2}{11!}Ax^{11} + \left(-\frac{4}{12!}A + \frac{1}{239500800}\right)x^{12} + \cdots,$$

$$\vdots$$

The series solution is given as:

$$y(x)=1+A\,x+\frac{1}{2!}x^2+\frac{1}{3!}B\,x^3+\frac{1}{4!}x^4+\frac{1}{5!}C\,x^5+\frac{1}{6!}x^6+\frac{1}{7!}Dx^7+\frac{1}{8!}x^8+\frac{1}{9!}Ex^9+\frac{1}{10!}Dx^{10}+\left(-\frac{1}{19958400}A+\frac{1}{39916800}\right)x^{11}$$
$$+\left(-\frac{1}{119750400}A+\frac{1}{159667200}\right)x^{12}+O(x^{13}),$$

Where

$$A=y'(0),B=y^{(3)}(0),\qquad C=y^{(5)}(0),\qquad D=y^{(7)}(0),\qquad E=y^{(9)}(0).$$

Imposing the boundary conditions at x = 1, we obtain

$$A=1.00001436,\qquad B=0.999858964,\quad C=1.001365775,$$
$$D=0.987457318,\qquad E=1.0932797434.$$

The series solution is given as:

$$y(x)=1+1.00001436\,x+\frac{1}{2!}x^2+0.1666431607x^3+\frac{1}{4!}x^4+0.008344714791x^5+\frac{1}{6!}x^6+0.00019524071x^7+\frac{1}{8!}x^8$$
$$+3.013\times10^{-6}x^9+\frac{1}{10\,!}x^{10}+2.51\times10^{-8}x^{11}-2.087\times10^{-9}x^{12}+\cdots.$$

Table 3.1: Exhibits the exact solution and the series solution along with the errors obtained by using the MDM

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	1.105170918	1.10517233	-1.41 E-6
0.2	1.221402758	1.221405446	-2.69 E-6
0.3	1.349858808	1.349862509	-3.70 E-6
0.4	1.491824698	1.49182905	-4.35 E-6
0.5	1.648721271	1.648725849	-4.58 E-6
0.6	1.822118800	1.822123158	-4.36 E-6
0.7	2.013752707	2.013756415	-3.71 E-6
0.8	2.225540928	2.225543623	-2.69 E-6
0.9	2.459603111	2.459604528	-1.42 E-6
1.0	2.718281828	2.7182830	2.00E-9

*Error=Exact solution-Series solution.

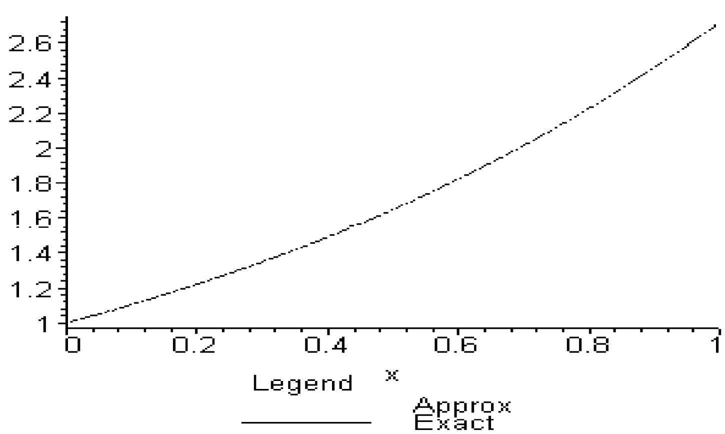

Fig. 1: Clearly indicates the accuracy of the proposed MDM.

Example 3.2: Consider the following linear boundary value problem of tenth-order

$$y^{(x)}(x) = -8e^{x} + y''(x)\,, \quad 0 < x < 1,$$

with boundary conditions

$$y(0) = 1,\ \ y'(0) = 0,\ \ y''(0) = -1,\ \ y'''(0) = -2,\ \ y^{(iv)}(0) = -3,$$
$$y(1) = 0,\ \ y'(0) = -e,\ \ y''(1) = -2e,\ \ y'''(1) = -3e,\ \ y^{(iv)}(1) = -4e,$$

The exact solution of the problem is

$$y(x) = (1-x)e^{x}.$$

Applying the modified decomposition method (MDM), we get

$$y_{n+1}(x) = y_n(x) + \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left(-8e^{x} + y_n''(x)\right) dx\,dx\,dx\,dx\,dx\,dx\,dx\,dx\,dx\,dx.$$

Consequently, following approximants are obtained

$$y_0(x) = c,$$
$$y_0(x) = 1,$$
$$y_1(x) = N\,y_0(x),$$
$$y_1(x) = -8e^{x} + 8 + 8x + \frac{7}{2!}x^2 + x^3 + \frac{5}{4!}x^4 + \left(\frac{1}{15} + \frac{1}{5!}A\right)x^5 + \left(\frac{1}{90} + \frac{1}{6!}B\right)x^6 + \left(\frac{1}{630} + \frac{1}{7!}C\right)x^7 + \left(\frac{1}{7!} + \frac{1}{8!}D\right)x^8 + \left(\frac{1}{45360} + \frac{1}{9!}E\right)x^9,$$
$$y_2(x) = N\left(y_0(x) + y_1(x)\right) - N\,y_0(x),$$

$$y_2(x) = \frac{1}{518400}x^{10} + \frac{1}{6652800}x^{11} + \frac{1}{95800320}x^{12} + \left(\frac{1}{778377600} + \frac{1}{6227020800}A\right)x^{13} + \cdots,$$

$\vdots$

The series solution is given by

$$y(x) = 17 - 16e^{x} + 16x + \frac{15}{2!}x^2 + \frac{7}{3}x^3 + \frac{13}{4!}x^4 + \left(\frac{2}{15} + \frac{1}{15!}\right)x^5 + \left(\frac{1}{45} + \frac{1}{6!}B\right)x^6 + \left(\frac{1}{315} + \frac{1}{7!}C\right)x^7 + \left(\frac{2}{7!} + \frac{1}{8!}D\right)x^8$$
$$+ \frac{1}{9!}(8+E)x^9 + \frac{7}{10!}x^{10} + \frac{6}{11!}x^{11} + \frac{5}{12!}x^{12} + \cdots.$$

Imposing the boundary conditions at x = 1 yields

$$A = -4.00002, \qquad B = -4.99999999, \qquad C = -6.00100, \qquad D = -7.00000, \qquad E = -8.010000.$$

The series solution is given by

$$y(x) = 17 - 16e^{x} + 16x + \frac{15}{2!}x^2 + \frac{7}{3}x^3 + \frac{13}{4!}x^4 + 0.99999999999997x^5 + 0.15277791666666666x^6 + 0.00198392857142857x^7 +$$
$$0.000233214285714286x^8 - 2.75573192239859\times 10^{-8}x^9 + \frac{7}{518400}x^{10} + \frac{1}{6652800}x^{11} + \frac{1}{9580320}x^{12} + \cdots.$$

Table 3.2: Exhibits the exact solution and the series solution along with the errors obtained by using the MDM.

x	Exact solution	Series solution	*Errors
0.00	1.0000000000	1.0000000000	0.0000000000
0.10	0.9946538263	0.9946538263	0.0000000000
0.20	0.9771222065	0.9771222065	0.0000000000
0.30	0.9449011653	0.9449011648	0.0000000005
0.40	0.8950948186	0.8950948125	00.0000000061
0.50	0.8243606354	0.8243605909	0.0000000444
0.60	0.7288475202	0.7288472928	0.0000002274
0.70	0.6041258122	0.6041249068	0.0000009055
0.80	0.4451081857	0.4451051850	0.0000030006
0.90	0.2459603111	0.2459516717	0.0000086394
1.00	0.0000000000	-.0000222586	0.0000222586

*Error =Exact solution- series solution

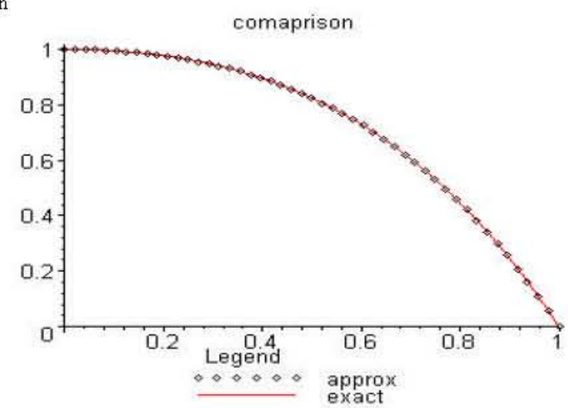

Fig. 2: clearly indicates the accuracy of the proposed MDM.

Example 3.3: Consider the following ninth order boundary value problem

$$y^{(ix)} = -9e^x + y(x), \quad 0 < x < 1$$

with boundary conditions

$$y(0) = 1, \;\; y^{(1)}(0) = 0, \; y^{(2)}(0) = -1, \; y^{(3)}(0) = -2, \;\; y^{(4)}(0) = -3,$$
$$y(1) = 0, \;\; y^{(1)}(0) = -e, \; y^{(2)}(1) = -2e, \; y^{(3)}(1) = -3e,$$

The exact solution of the problem is

$$y(x) = (1 - x)e^x$$

Applying the modified decomposition method (MDM), we get

$$y_{n+1}(x) = y_n(x) + \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left(-9e^x + y_n(x)\right) dx\,dx\,dx\,dx\,dx\,dx\,dx\,dx\,dx.$$

Consequently, following approximants are obtained

$y_0(x) = c,$

$y_0(x) = 1,$

$y_1(x) = N\,y_0(x),$

$$y_1(x) = -\frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}Ax^5 + \frac{1}{6!}Bx^6 + \frac{1}{7!}Cx^7 + \frac{1}{8!}Dx^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \cdots,$$

$\vdots$

The series solution is given by

$$y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}Ax^5 + \frac{1}{6!}Bx^6 + \frac{1}{7!}Cx^7 + \frac{1}{8!}Dx^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \cdots.$$

Imposing the boundary condition at x=1 gives

$$A = -3.999992, \quad B = -5.00017, \quad C = -5.9985, \quad D = -7.005.$$

The series solution is given as

$$y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - 0.03333326667x^5 - 0.006944680556x^6 - 0.001190178571x^7 - 0.000173735119x^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \cdots.$$

Table 3.3: Exhibits the exact solution and the series solution along with the errors obtained by using the MDM.

xx	Exact solution	Series solution	*Errors
0.0	1.00000000	1.0000000000	0.000000
0.1	0.99465383	0.9946538264	-2.0E-10
0.2	0.97712221	0.9771222066	-2.0E-10
0.3	0.94490117	0.9449011654	-2.0E-10
0.4	0.89509482	0.8950948186	-2.0E-10
0.5	0.82436064	0.8243606355	-2.0E-10
0.6	0.72884752	0.7288475206	-6.0E-10
0.7	0.60412581	0.6041258131	-1.0E-9
0.8	0.44510819	0.4451081876	-2.0E-9
0.9	0.24596031	0.2459603145	-3.4E-9
1.0	0.00000000	0.0000000000	0.000000

*Error=Exact solution-Series solution.

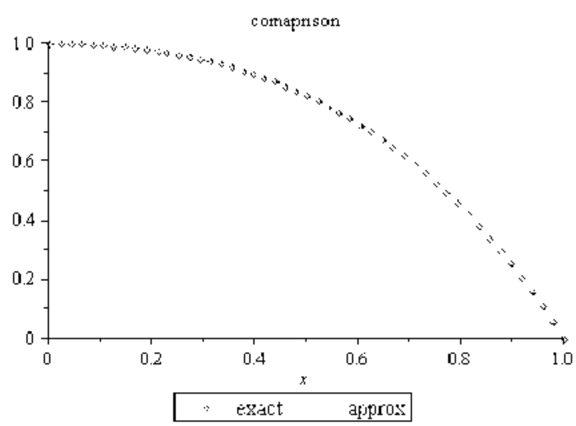

Fig. 3: clearly indicates the accuracy of the proposed MDM.

CONCLUSIONS

In this paper, we applied the modified decomposition method (NDM) for finding the solution of ninth and tenth-order boundary value problems. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. It may be concluded that MDM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result. The fact that the MDM solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

REFERENCES

1. Abbasbandy, S., 2007. Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method, Internat. J. Numer. Mech. Engg., 70: 876-881.
2. Abbasbandy, S., 2007. A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials, J. Comput. Appl. Math., 207: 59-63.
3. Abdou M.A. and A.A. Soliman, 2005. New applications of variational iteration method, Phys. D, 211(1-2): 1-8.
4. Chandrasekhar, S., 1981. Hydrodynamic and hydro magnetic stability, Dover, New York.
5. Djidjeli, K., E.H. Twizell and A. Boutayeb, 1993. Numerical methods for special nonlinear boundary value problems of order 2m, J. Comput. Appl. Math. 47: 35-45.
6. Geijji, V.D. and H. Jafari, 2006. An iterative method for solving nonlinear functional equations, J. Math. Anal. Appl., 316: 753-763.
7. He, J.H., 2006. Homotopy perturbation method for solving boundary value problems, Phys. Lett. A, 350: 87-88.
8. He, J.H., 2007. The variational iteration method for eighth-order initial boundary value problems, Phys. Scr., 76(6): 680-682.
9. Mohyud-Din, S.T. and M.A. Noor, 2007. Homotopy perturbation method for solving fourth-order boundary value problems, Math. Prob. Engg., 1-15: Article ID 98602, doi:10.1155/2007/98602.
10. Noor, M.A. and S.T. Mohyud-Din, 2008. Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials, Int. J. Nonlin. Scne. Num. Simul., 9(2).
11. Noor, M.A. and S.T. Mohyud-Din, 2008. Solution of twelfth-order boundary value problems by variational iteration technique, J. Appl. Math. Computg., DOI: 10.1007/s12190-008-0081-0.
12. Noor, M.A. and S.T. Mohyud-Din, 2008. Homotopy perturbation method for nonlinear higher-order boundary value problems, International J. Nonlinear Sciences and Numerical Simulation, 9(4): 395-408.
13. Mohyud-Din, S.T. M.A. Noor and K.I. Noor, 2009. Some relatively new techniques for nonlinear problems, Mathematical Problems in Engineering, Hindawi, (2009); Article ID 234849, 25 pages, doi:10.1155/2009/234849.
14. Mohyud-Din, S.T. and M.A. Noor, 2009. Homotopy perturbation method for solving partial differential equations, Zeitschrift für Naturforschung A-A J. Physical Sci., 64a: 157-170.
15. Mohyud-Din, S.T., 2009. Solution of nonlinear differential equations by exp-function method, World Appl. Sci. J., 7: 116-147.
16. Noor, M.A. and S.T. Mohyud-Din, 2007. Variational iteration technique for solving higher order boundary value problems, Appl. Math. Comput., 189: 1929-1942.
17. Noor, M.A. and S.T. Mohyud-Din, 2007. An efficient method for fourth order boundary value problems, Comput. Math. Appl., 54: 1101-1111.
18. Mohyud-Din, S.T. and A. Yildirim, 2010. An iterative algorithm for fifth-order boundary value problems, World Appl. Sci. J., 8(5): 531-535.
19. Mohyud-Din, S.T., A. Yildirim and M.M. Hosseini, 2010. An iterative algorithm for time-fractional Navier–Stokes equations, World Appl. Sci. J.,
20. Noor, M.A., K.I. Noor, S.T. Mohyud-Din and A. Shabir, 2006. An iterative method with cubic convergence for nonlinear equations, Appl. Math. Comput., 183: 1249-1255.
21. Noor, M.A. and S.T. Mohyud-Din, 2007. An iterative method for solving Helmholtz equations, A. J. Math. Mathl. Sci., 1: 9-15.
22. Mohyud-Din, S.T., 2010. Variational iteration techniques for boundary value problems, VDM Verlag, ISBN 978-3-639-27664-0.
23. Wazwaz, A.M., 2000. The modified decomposition method for solving linear and nonlinear boundary value problems of tenth-order and twelfth-order, Int. J. Nonlin. Sci. Num. Sim., 1: 17-24.