

Breaking Soliton Equation and Travelling Wave Solutions

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Abstract: In the present paper, we construct the travelling wave solutions involving parameters of the breaking soliton equation. When the parameters are taken as special values the solitary waves are also derived from the travelling waves. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions.

Key words: Breaking soliton equation • $(\frac{G'}{G})$ -expansion method

INTRODUCTION

In this study we consider the breaking soliton equation in the following form.

$$u_{tt} - u_{xxx} - 1u_{xx}u_y - 4u_xu_{xy} = 0$$

And seeking for construct new exact solutions for this equation. Recently, interest has increased in traveling wave solutions of differential equations. There are many methods to solve these equations in literature. Some of them are: the Hirota's bilinear method [1, 2], the generalized Riccati equation [3], the Weierstrass elliptic function method [4], the theta function method [5,6], the sine-cosine method [7], the Jacobi elliptic function expansion [8,9], the complex hyperbolic function method [10,11], the sub-ODE method [12,13] and so on. In the present Letter we shall propose a direct and concise method which is called the $(\frac{G'}{G})$ -expansion method [14]

to look for travelling wave solutions of nonlinear evolution equations. The $(\frac{G'}{G})$ -expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $(\frac{G'}{G})$, and

that $G = G(\xi)$ satisfies a second order linear ordinary differential equation (O.D.E).

Description of the $(\frac{G'}{G})$ -Expansion Method: For doing

this we considering the nonlinear partial differential equation in the form

$$P(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots) = 0 \quad (1)$$

Combining the independent variables x and t into one we suppose that $\xi = x - vt$, variable

$$u(x, t) = u(\xi), \quad \xi = x - vt \quad (2)$$

The travelling wave variable (2) permits us to reduce namely $G = G(\xi)$, Eq.(1) to an O.D.E. for

$$P(u, -vu', u', v^2u'', -vu'', u'', \dots) = 0 \quad (3)$$

suppose that the solution of O.D.E. (3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows

$$u(\xi) = \alpha_m \left(\frac{G'}{G} \right) + \dots, \quad (4)$$

Where $G = G(\xi)$ satisfies the second order L.O.D.E. in the form

$$G'' + \lambda G' + \mu G = 0 \quad (5)$$

α_m, \dots, λ and μ are constants to be determined later $\alpha_m \neq 0$, the unwritten part in 4 is also a polynomial in $(\frac{G'}{G})$, but

the degree of which is generally equal to or less than $m - 1$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in O.D.E. (3). By substituting (4) into Eq. (3) and using the second order linear O.D.E. (5), collecting all terms with the same order $(\frac{G'}{G})$ together, the left-hand side of Eq. (3) is

converted into another polynomial in $(\frac{G'}{G})$ - Equating

each coefficient of this polynomial to zero yields a set of algebraic equations for α_m, \dots, λ and u . Assuming that the constants α_m, \dots, λ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (5) have been well known for us, then substituting α_m, \dots, v and the general solutions of Eq. (5) into (4) we have more travelling wave solutions of the nonlinear evolution equation (1).

Application for Breaking Soliton Equation: In this section, we apply the $(\frac{G'}{G})$ -expansion method to construct the traveling wave solutions for breaking soliton equation in following form

$$u_{xx} - u_{xxx} - 2u_{xx}u_y - 4u_xu_{xy} = 0 \quad (6)$$

The travelling wave variable below

$$u(x, t) = u(\xi) \quad \xi = x + y - \omega t \quad (7)$$

Permits us converting Eq.(7) into an ODE for $G = G(\xi)$ we have

$$-\omega u'' - u''' - 6u''u' = 0$$

Integrating it with respect to ξ once yields

$$-\omega u' + u'' + 2(u')^2 + c = 0$$

Where c is integration constant. Suppose that the solution of ODE (8) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_m (\frac{G'}{G}) + \dots, \quad (9)$$

Where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \quad (10)$$

$\alpha_m, \dots, \alpha_1, \alpha_0$ and μ are to be determined later. By using (9) and (10) and considering the homogeneous balance we required that $m = 2$. So we can write (9) as

$$u(\xi) = \alpha_2 (\frac{G'}{G})^2 + \alpha_1 (\frac{G'}{G}) + \alpha_0 \quad (11)$$

By substituting (11) into Eq. (8) and collecting all terms with the same power of $(\frac{G'}{G})$ together, yields a set of simultaneous algebraic equations for $\alpha_2, \alpha_1, \alpha_0, v, \lambda, \mu$ and c as follows:

$$6\alpha_2(2\alpha_2\lambda + \alpha_1) = 0$$

$$6\alpha_2(2\alpha_2\mu + \alpha_1\lambda) + 6\alpha_2 + 3(2\alpha_2\lambda + \alpha_1)^2 = 0$$

$$6\alpha_1\alpha_2\mu + (2\alpha_1 + 10\alpha_2\lambda) + 2\omega\alpha_2 = 0$$

$$3(2\alpha_2\mu + \alpha_1\lambda)^2 + 3\alpha_1\mu(2\alpha_2\lambda + \alpha_1) + (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda) + \omega(2\alpha_2\lambda + \alpha_1) = 0$$

$$3\alpha_2\mu(2\alpha_2\mu + \alpha_1\lambda) + (6\alpha_2\lambda + 2\alpha_1\mu + \alpha_1\lambda^2) + \omega(2\alpha_2\mu + \alpha_1\lambda) = 0$$

$$3(\alpha_1\mu)^2 + 2\alpha_2\mu^2 + \alpha_1\lambda\mu + \omega\alpha_1\mu + c = 0$$

Solving algebraic equations above by Maple package yields

$$\begin{aligned} \alpha_2 &= -\frac{1}{2(\mu - \lambda^2)}, & \alpha_1 &= \frac{\lambda}{\mu - \lambda^2} \\ \omega &= -\frac{3\lambda(2\mu - \lambda^2)}{\mu - \lambda^2} \\ c &= \frac{\mu(-5\lambda^2\mu + \mu^2 + \lambda^4 + \omega\lambda\mu - \omega\lambda^3)}{(\mu - \lambda^2)^2} \end{aligned}$$

Substituting relations above into equation (11) we have

$$u(\xi) = -\frac{1}{2(\mu - \lambda^2)} (\frac{G'}{G}) + \frac{\lambda}{\mu - \lambda^2} (\frac{G'}{G}) \quad (12)$$

And $\xi = x + \frac{3\lambda(2\mu - \lambda^2)}{\mu - \lambda^2} t$. Substituting the general solutions of Eq. (10) as follows

$$\frac{G'}{G} = \frac{1}{2} \sqrt{\lambda^2 - 4\mu}$$

$$\times \left(\frac{C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right) - \frac{\lambda}{2}$$

Into (12) we have three types of travelling wave solutions of the breaking soliton equation (6) as follows:

When $\lambda^2 - 4\mu > 0$

$$u(\xi) = -\frac{\lambda^2 - 4\mu}{8(\mu - \lambda^2)}$$

$$\times \left(\frac{C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right)$$

$$+ \frac{2\lambda - \lambda(\mu - \lambda^2)}{2(\mu - \lambda^2)}$$

Where $\xi = x + \frac{3\lambda(2\mu - \lambda^2)}{\mu - \lambda^2}t$. C_1 and C_2 are arbitrary

constants.

In particular, if $c_1 > 0, c_1^2 > c_2^2$, then $u = u(\xi)$ can be written as

$$u(\xi) = -\frac{\lambda^2 - 4\mu}{8(\mu - \lambda^2)} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + \xi_0 \right)$$

$$+ \frac{2\lambda - \lambda(\mu - \lambda^2)}{2(\mu - \lambda^2)}$$

Which is the solitary wave solution of the breaking soliton equation.

When $\lambda^2 - 4\mu < 0$

$$u(\xi) = -\frac{4\mu - \lambda^2}{4(\mu - \lambda^2)}$$

$$\times \left(\frac{-C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right)$$

$$+ \frac{2\lambda - \lambda(\mu - \lambda^2)}{2(\mu - \lambda^2)}$$

When $\lambda^2 - 4\mu = 0$

$$u(\xi) = \frac{-C_2^2}{2(\mu - \lambda^2)(C_1 + C_2 \xi)^2} + \frac{2\lambda - \lambda(\mu - \lambda^2)}{2(\mu - \lambda^2)}$$

$$\xi = x + \frac{3\lambda(2\mu - \lambda^2)}{\mu - \lambda^2}t$$

Where C_1 and C_2 are arbitrary constants.

CONCLUSION

The solutions of these non-linear evolution equations have many potential applications in physics. These equations are very difficult to be solved by traditional methods.

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