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Application of Semi-Analytical Homotopy Perturbation Method on Some System of Nonlinear Integral-Differential Equations

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Abstract: In this study Homotopy perturbation method (HPM) has been applied to solve the system of nonlinear integral-differential equations. We compared our solution with the exact solution. Some examples are given to illustrate the ability and reliability of the method and comparison of them against the exact solution is presented. The results reveal that the method is very simple and effective in solving nonlinear integral-differential equations.

Key words: Homotopy perturbation method · System of nonlinear integral-differential equations · Exact solution

INTRODUCTION

In this study, HPM has been used for solving the system of nonlinear integral-differential equations. To support our work three nonlinear integral-differential equations are provided. Recently Biazar and Aminikhah [1] used Variational iteration method (VIM) for solving the governing problem. The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji-Huan He [2-4]. This technique has been employed to solve a large variety of linear and nonlinear problems [5-15].

Basic Idea of He's Homotopy Perturbation Method: To illustrate the basic ideas of this method, we consider the following nonlinear differential Equation:

$$A(u) - f(r) = 0, r \in \Omega \tag{1}$$

Considering the boundary conditions of:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \tag{2}$$

Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be, generally divided into two parts of L and N, where L is the linear part, while N is the nonlinear one. Eq. (7) can, therefore, be rewritten as:

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

By the homotopy technique, we construct a homotopy as $v(r,p)$: $\Omega\times[0,1]\neg R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] = p[A(v) - f(r)] = 0, p \in [0,1], r \in \Omega \tag{4}$$

Or

$$H(v,p) = L(v) - L(u_0) = pL(u_0) = p[N(v) - f(r)] = 0 \tag{5}$$

Where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (2) which satisfy the boundary conditions. Obviously, considering Eqs. (10) and (11), we will have:

$$H(v,p) = L(v) - L(u_0) = 0 \tag{6}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{7}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy.

According to HPM, we can first use the embedding parameter p as a "small parameter" and assume that the solution of Eqs. (10) and (11) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2v_2 + ... \tag{8}$$

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Setting p=1 results in the approximate solution of Eq.(7):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{9}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method, which lessens the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques.

The series (15) is convergent for most cases. However, the convergence rate depends on the nonlinear operator $A(v)$. The following opinions are suggested by He:

- The second derivative of $N(v)$ with respect to v must be small because the parameter p may be relatively large, i.e. $p \to 1$.
- The norm of $L^{-1}\partial N / \partial N$ must be smaller than one so that the series converges.

Example 1: Consider the following system of two nonlinear integral-differential equations [1] whose exact solution is $f(x) = x + e^x$, $g(x) = x - e^x$:

$$f''(x) = 1 - \frac{1}{3}x^3 - \frac{1}{2}g'^2(x) + \frac{1}{2}\int_0^x (f^2(t) + g^2(t))dt$$
$$g''(x) = -1 + x^2 - xf(x) + \frac{1}{4}\int_0^x (f^2(t) - g^2(t))dt \tag{10}$$

With the initial conditions:

$$f(0) = 1, f'(0) = 2$$
$$g(0) = -1, g'(0) = 0. \tag{11}$$

The homotopy of the above mentioned system could be constructed, using Homotopy perturbation method:

$$f''(x) = p[1 - \frac{1}{3}x^3 - \frac{1}{2}g'^2(x) + \frac{1}{2}\int_0^x (f^2(t) + g^2(t))dt]$$
$$g''(x) = p[-1 + x^2 - xf(x) + \frac{1}{4}\int_0^x (f^2(t) - g^2(t))dt] \tag{12}$$

Also the homotopy embedding parameter is used as follows:

$$f(x) = ,f_0(x) + pf_1(x) + p^2f_2(x) + p^3f_3(x) + ...$$
$$g(x) = ,g_0(x) + pg_1(x) + p^2g_2(x) + p^3g_3(x) + ... \tag{13}$$

Substituting Eqs. (13) into Eq. (12) and rearranging based on powers of p-terms, we have:

$$p^0 : f_0''(x) = 0, \quad f_0(0) = 1, \quad f_0'(0) = 2$$
$$p^1 : f_1''(x) = 1 - \frac{1}{3}x^3 - \frac{1}{2}(g_0'(x))^2 + \frac{1}{2}\int_0^x (f_0^2(t) + g_0^2(t))dt, \quad f_1(0) = 0, \quad f_1'(0) = 2$$
$$p^2 : f_2''(x) = -\frac{1}{2}(2g_0'(x)g_1'(x)) + \frac{1}{2}\int_0^x (2f_0(t)f_1(t) + 2g_0(t)g_1(t))dt, \quad f_2(0) = 0, \quad f_2'(0) = 0,$$
$$\vdots \tag{14}$$
$$p^0 : g_0''(x) = 0, \quad g_0(0) = -1, \quad g_0'(0) = 0$$
$$p^1 : g_1''(x) = -1 + x^2 - xf_0(x) + \frac{1}{4}\int_0^x (f_0^2(t) - g_0^2(t))dt, \quad g_1(0) = 0, \quad g_1'(0) = 0,$$
$$p^2 : g_2''(x) = -xf_1(x) + \frac{1}{2}(2g_0'(x)g_1'(x)) + \frac{1}{4}\int_0^x (2f_0(t)f_1(t) + 2g_0(t)g_1(t))dt, \quad g_2(0) = 0, \quad g_2'(0) = 0,$$
$$\vdots$$

Solving the above equations using the initial conditions yield:

$$f_0(x) = f(0) + xf_0'(x) = 1 + 2x$$
$$g_0(x) = g(0) + xg_0'(x) = -1$$
$$f_1(x) = 0.5x^2 + 0.1666666667x^3 + 0.0833333333x^4 + 0.0166666666x^5 \tag{15}$$
$$g_1(x) = -0.5x^2 - 0.1666666667x^3 - 0.0416666666x^4 + 0.0166666666x^5$$
$$..............$$

In this manner the other components can be easily obtained.

Here, setting p= 1 in Eq. (13), we obtain the following fifth-order approximation solutions of system of two nonlinear integral-differential Eqs.(10):

$$f(x) \approx f_0(x) + f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x)$$
$$g(x) \approx g_0(x) + g_1(x) + g_2(x) + g_3(x) + g_4(x) + g_5(x) \tag{16}$$

The accuracy of the HPM for the system of two nonlinear integral-differential equations is controllable and absolute errors are very small. These results are listed in Table 1.

Example 2: consider the following system of two nonlinear integral-differential equations [1] exact solution is $f(x) = \sinh x$, $g(x) = \cosh x$:

$$f'(x) = 1 - \frac{1}{2}g'^2(x) + \int_0^x ((x-t)f(t) + f(t)g\ (t))dt$$
$$g'(x) = 2x + \int_0^x ((x-t)f(t) - g^2(t) + f^2(t))dt \qquad (17)$$

With the boundary conditions:

$$f(0) = 1,$$
$$g(0) = 1. \qquad (18)$$

So the following homotopy of the Eq.(9) is constructed according to homotopy perturbation method:

$$f'(x) = p[1 - \frac{1}{2}g'^2(x) + \int_0^x ((x-t)f(t) + f(t)g\ (t))dt]$$
$$g'(x) = p[2x + \int_0^x ((x-t)f(t) - g^2(t) + f^2(t))dt] \qquad (19)$$

In view of the HPM, we use the homotopy parameter p to expand solution,

$$f(x) = f_0(x) + pf_1(x) + p^2 f_2(x) + p^3 f_3(x)$$
$$g(x) = g_0(x) + pg_1(x) + p^2 g_2(x) + p^3 g_3(x) \qquad (20)$$

Substituting (20) into (19) and equating the coefficients of like powers of p, we get the following set of differential equations:

$$p^0 : f_0'(x) = 0, \quad f_0(0) = 0,$$
$$p^1 : f_1'\ (x) = 1 - \frac{1}{2}(g_0'(x))^2 + \int_0^x ((x-t)f_0\ (t) +$$
$$f_0\ (t)g_0\ (t))dt, \quad f_1(0) = 0, \qquad (21)$$
$$p^2 : f_2'\ (x) = -2g_0'(x)g_1'(x) + \int_0^x ((x-t)f_1\ (t) +$$
$$2f_0(t)g_1(t) + f_1\ (t)g_0(t))dt, \quad f_2(0) = 0,$$

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$$p^0 : g_0'(x) = 0, \quad g_0(0) = 1,$$
$$p^1 : g_1'(x) = 2x + \int_0^x ((x-t)f_0\ (t) - g_0^2(t) +$$
$$g_0^2(t))dt, \quad g_1(0) = 0,$$
$$p^2 : g_2'(x) = \int_0^x ((x-t)f_1(t) + 2g_0(t)g_1(t) +$$
$$2f_0\ (t)f_1(t))dt, \quad g_2(0) = 0,$$

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Solving the above equations using the initial conditions yield

Table 1: Numerical values of solution of Example 1

X	f(x)[exact solution]	f(x)[HPM]	|f(x)[exact]-f(x)[HPM]|	g(x)[exact solution]	g(x)[HPM]	|g(x)[exact] -g(x)[HPM]
0	1	1	0	-1	-1	0
0.1	1.2051709180756476248	1.2051709180756822582	3.9968E-14	-1.0051709180756476248	-1.0051709180756823583	03.9968E-14
0.2	1.4214027581601698339	1.4214027581604567666	2.8999E-13	-1.0214027581601698339	-1.0214027581604599660	2.8999E-13
0.3	1.6498588075760031040	1.6498588075770051236	1.00009E-12	-1.0498588075760031040	-1.0498588075770270443	1.02007E-12
0.4	1.8918246976412703178	1.8918246976438453853	2.56994E-12	-1.0918246976412703178	-1.0918246976437964880	2.51998E-12
0.5	2.1487212707001281468	2.1487212707082564905	8.12994E-12	-1.1487212707001281468	-1.1487212707048470009	4.72E-12
0.6	2.4221188003905089749	2.4221188004425423404	5.20401E-11	-1.2221188003905089749	-1.2221188003920284340	1.52012E-12
0.7	2.7137527074704765216	2.7137527078744611183	4.0399E-10	-1.3137527074704765216	-1.3137527074018120704	6.866E-11
0.8	3.0255409284924676046	3.0255409311263433743	2.63388E-09	-1.4255409284924676046	-1.4255409278571336394	6.3533E-10
0.9	3.3596031111569496638	3.3596031251403591867	1.39834E-08	-1.5596031111569496638	-1.5596031071210196390	4.03593E-09
1	3.7182818284590452354	3.7182818906515237858	6.21925E-08	-1.7182818284590452354	-1.7182818076689979447	2.079E-08

Table 2 Numerical values of solution of Example 2

X	*f(x)[exact solution]*	*f(x)[HPM]*	|*f(x)[exact]-f(x)[HPM]*|	*g(x)[exact solution]*	*g(x)[HPM]*	|*g(x)[exact] -g(x)[HPM]*|
0	0	0	0	1	1	0
0.1	0.10016675001984402582	0.10016675001984451110	0	1.0050041680558035990	1.0050041680558035829	0
0.2	0.20133600254109398763	0.20133600254210191841	0-1.008E-12	1.0200667556190758463	1.0200667556190087092	0-6.99441E-14
0.3	0.30452029344714261896	0.30452029353639097687	-8.9248E-11	1.0453385141288604850	1.0453385141200327774	-8.83005E-12
0.4	0.41075232580281550854	0.41075232798614418927	-2.18333E-09	1.0810723718384548093	1.0810723715545191347	-2.8394E-10
0.5	0.52109530549374736162	0.52109533199769562790	-2.65039E-08	1.1276259652063807852	1.1276259609741575227	-4.23223E-09
0.6	0.63665358214824127112	0.63665378936509788580	-2.07217E-07	1.1854652182422677038	1.1854651793825052791	-3.88598E-08
0.7	0.75858370183953350346	0.75858490089827381958	-1.19906E-06	1.2551690056309430182	1.2551687497970409435	-2.55834E-07
0.8	0.88810598218762300658	0.88811156125378129805	-5.57907E-06	1.3374349463048445980	1.3374336240481582055	-1.32226E-06
0.9	1.02651672570817527600	1.02653874348391543640	-2.20178E-05	1.4330863854487743878	1.4330806971329216755	-5.68832E-06
1	1.17520119364380145690	1.17527762340136791760	-7.64298E-05	1.5430806348152437785	1.5430594351845813450	-2.11996E-05

$$
\begin{aligned}
&f_0(x) = f(0) = 0\\
&g_0(x) = g(0) = 1\\
&f_1(x) = x + 0.166666667x^3\\
&g_1(x) = 0.5x^2 \qquad (22)\\
&f_2(x) = 0.04166666666x^5 + 0.001984126984x^7\\
&g_2(x) = 0.0416666666x^4 + 0.0416666667x^6 +\\
&0.000496031x^8
\end{aligned}
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In this manner, the rest of the components of the homotopy perturbation series can be obtained. Here, setting $p= 1$ in Eq. (20), we obtain the following fifth-order approximation solutions of system of two nonlinear integral-differential Eqs.(8):

$$f(x) \approx f_0(x) + f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x)$$
$$g(x) \approx g_0(x) + g_1(x) + g\,(x) + g\,(x) + g\,(x) + g\,(x) \qquad (23)$$

The accuracy of the HPM for the system of two nonlinear integral-differential equations is controllable and absolute errors are very small. These results are listed in Table 2.

CONCLUSIONS

In this study, Homotopy perturbation method (HPM) has been applied to find the approximate solution to the system of nonlinear integral-differential equations. The results obtained have been compared against the exact solution. It may be concluded that homotopy perturbation methodology is a very powerful and efficient technique in finding exact and approximate solutions for wide variety of problems of great importance in engineering and sciences. Considering the traditional perturbation method it could be concluded that, as HPM does not need discretization of the variable, i.e., time and space, it is not affected by computation round off error and necessity of large computer and time.

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