World Applied Sciences Journal 10 (7): 803-808, 2010
ISSN 1818-4952

Bi Laplace Transform of a Product of Some Special Function

Ahmad Ali Al-Jarrah

Department of Mathematics and Statistics, Mu'tah University, P.O.Box 25, Mu'tah, Al-Karak, Jordan

Abstract: Our aim in this paper is to obtain two infinite double integral concerning to exponential functions, the product of the Gauss hypergeometric function [1], Fox's H-function [2], a general class of polynomials [3], the polynomial set [4] and the H-function of two variables [5]. Some interesting cases are also given.

Key words: H-function · Gauss Hypergeometric function · Laplace transform

INTRODUCTION

The two dimensional Laplace transform of a function $f(x_1,x_2)$ will be defined as

$$L\{f(x_1, x_2):p_1,p_2\} = \int_0^\infty\int_0^\infty e^{-px_1-p2x_2} f(x_1, x_2)dx_1\, dx_2, \tag{1.1}$$

The generalized polynomial set $S_v^{\lambda,\mu,\rho}$ [x] is defined by the following Rodrigues type formula [4]

$$S_v^{\lambda,\mu,\rho}[x:t,p,\varpi,C,D,U,u,v]$$

$$=(Cx+D)^{-\lambda}\ (1-\rho x^t)^{-\mu/\rho}\ T_{u,v}^{U+V}[Cx+d)^{\lambda+\omega V}(1-\rho x^t)^{\frac{u}{t}}+\rho v], \tag{1.2}$$

Where the differential operator $T_{u,v}$ is defined as

$$T_{u,v} \equiv x^v\left[u + x\frac{d}{dx}\right] \tag{1.3}$$

The explicit form of this generalization polynomial set [4] is given by

$$S_v[x:t,p,\omega,C,D,U,u,v] = D^{\omega V}x^{v(U+V)}(1-\rho x^t)^{\rho V}\ v^{U+V}\sum_{\ell=0}^{U+V}\sum_{k=0}^{\ell}\sum_{j=0\lambda,\mu i\neq 0}^{U+V}\sum^{j}\frac{(-1)^j(-j)_i(\lambda)_j(-\ell)_k}{(1-\lambda-j)_i\ \ell!\,k!\,i!\,j!}$$

$$.(-\lambda-\omega V)_i\left(-\frac{\mu}{\rho}-pV\right)\left(\frac{i+u+pk}{v}\right)_{U+V}\left(-\frac{\rho x^t}{1-\rho x^t}\right)^{\ell}\left(\frac{Cx}{D}\right)^i \tag{1.4}$$

If we take C = 1, D = 0 and ρ → 0 in (1.3) and using the well known confluence principle

$$\lim_{|b|\to\infty}(b)_V\left(\frac{x}{b}\right)^V = x^V$$

Corresponding Author: Ahmad Ali Al-Jarrah, Department of Mathematics and Statistics, Mu'tah University, P.O.Box 25, Mu'tah, Al-Karak, Jordan E-mail: taqwa22000@yahoo.com

There in, we have the following polynomial set

$$S_v^{\lambda,\mu,0}[x] = S_V^{\lambda,\mu,0}[x : t, p, w, 1, 0, U, u, v]$$

$$= x^{\omega V + v(U+V)} \sum_{\ell=0}^{U+V} \sum_{k=0}^{\ell} \frac{(-\ell)_k}{k!\,\ell!} \left(\frac{\lambda + \omega V + u + tk}{v}\right)_{U+V} v^{U+V} (\mu x^t)^{\ell} \tag{1.5}$$

Srivastava [11] introduced the general class of polynomials

$$S_n^m[x] = \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n,s} x^s, \quad s = 0,1,2,..... \tag{1.6}$$

Where m is an arbitrary positive integer and the coefficients $A_{n,s}$ ($n,s \geq 0$) are arbitrary constants, real or complex. The H-function of two variables defined by [7], possesses the following integral representation

$$H[x,y] = H^{0,N:(M_1,N_1);(M_2,N_2)}_{P,Q:(P_2,Q_2);\,(P_2,Q_2)}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j:\alpha_j',\alpha_j'')_{1,P}:(c_j'\gamma_j')_{1,P_1};(c_j'',\gamma_j'')_{1,P_2} \\ (bj:\beta',\beta'')_{1,Q}:(d_j',\delta_j')_{1,Q_1};(d_j'',\delta_j'')_{1,Q_2} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \theta(\xi,\eta)\theta_1(\xi)\theta_2(\eta) x^{\xi} y^{\eta}\, d\xi\, d\eta\,, \tag{1.7}$$

Where

$$\phi(\xi,\eta) = \frac{\prod_{j=1}^{n} \Gamma(1-a_j+\alpha_j'\xi+\alpha_j''\xi)}{\prod_{j=N+1}^{P} \Gamma(a_j-\alpha_j'\xi-\alpha_j''\xi) \prod_{j=1}^{Q'} \Gamma(1-b_j+\beta'\xi+\beta_j''\eta)}, \tag{1.8}$$

$$\theta_1(\xi) = \frac{\prod_{j=1}^{M_1} \Gamma(d_j'-\delta_j'\xi) \prod_{j=1}^{N_1} \Gamma(1-c_j+\gamma_j\xi)}{\prod_{j=M_1+1}^{Q_1} (1-d_j'+\delta_j'\xi) \prod_{j=N_1+1}^{P_1} \Gamma(c_j-\gamma_j'\xi)}, \tag{1.9}$$

and

$$\theta_2(\eta) = \frac{\prod_{j=1}^{M_2} \Gamma(d_j'-\delta_j''\eta) \prod_{j=1}^{N_1} \Gamma(1-c_j''+\gamma_j'\eta)}{\prod_{j=M_2+1}^{Q_2} \Gamma(1-d_j''+\delta_j''\eta) \prod_{j=N+1}^{P_2} \Gamma(c_j''-\gamma_j\eta)}. \tag{1.10}$$

For the convergence, existence condition and other details of the H-function of two variables we refer to [6, 7]. The series representation of Fox's H-function [8, 9].

$$H^{M',N'}_{P',Q'}\left[y\middle|\begin{matrix}(e_{P'},E_{P'})\\(f_{Q'},F_{Q'})\end{matrix}\right] = \sum_{G=0}^{\infty} \sum_{g=1}^{M} \frac{(-1)^G \phi(\eta_G) y^{\eta_G}}{G!\,F_g}, \tag{1.11}$$

Where

$$\phi(\eta_G) = \frac{\prod_{j=1,j\neq g}^{M'} \Gamma(f_j-F_j\eta_G) \prod_{j=1}^{N'} \Gamma(1-e_j+E_j\eta_G)}{\prod_{j=M'+1}^{Q'} \Gamma(1-f_j+F_j\eta_G) \prod_{j=N+1}^{P'} \Gamma(e_j-E_j\eta_G)}, \tag{1.12}$$

and $\eta_G = \frac{(f_g + G)}{F_g}$

The Gauss hypergeometric function [1, 10] is defined as

$$ {}_2F_1\left(\begin{matrix} a', b' \\ \rho' \end{matrix}; x_3\right) = \sum_{s'=0}^{\infty} \frac{(a')_{s'}(b')_{s'} x_3^{s}}{(\rho')_{s'}\, s'!} \tag{1.13} $$

for ρ' neither zero nor a negative integer and Re (ρ'- b' - a') > 0.

Main Integral Transformations

First Integral Transformation:

$$ \int_0^{\infty}\int_0^{\infty} (\alpha_1 y_1 + \alpha_2 y_2)^{h_1-1}(\beta_1 y_1 + \beta_2 y_2)^{h_2-1} $$

$$ .exp\,[-p_1(\alpha_1 y_1 + \alpha_2 y_2) - p_2(\beta_1 y_1 + \beta_2 y_2)] $$

$$.{}_2F_1\left(\begin{matrix} a', b' \\ \rho' \end{matrix}; x_3(\alpha_1 y_1 + \alpha_2 y_2)\right) S_n^m(\alpha_1, y_1 + \alpha_2, y_2) $$

$$ .S_V^{\lambda,\mu,O}(\beta_1 y_1 + \beta_2 y_2) H_{P',Q'}^{M',N'}[(\alpha_1 y_1 + \alpha_2 y_2)^{\sigma}]dy_1 dy_2 $$

$$ = \frac{1}{R}\sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n.s} \sum_{\ell=0}^{U+V}\sum_{k=0}^{\ell} \frac{(-\ell)_k v^{U+V}\mu^{\ell}}{k!\,\ell!} $$

$$.\left(\frac{\lambda + \omega V + u + tk}{v}\right)_{U+V} \sum_{s'=0}^{\infty} \frac{(a')_{s'}(b')_{s'} x_3^{s'}}{s'!\,(\rho')_{s'}} \frac{1}{p_1^{h_1+s+s'}\, p_2^{h_2+\omega V+v(U+V)+t\ell}} $$

$$ .H_{P'+1,Q'}^{M',N'+1}\left[p_1^{-\sigma} \left| \begin{matrix} (1-h_1-s-s';\sigma),(e_{P'},E_{P'}) \\ (f_{Q'},F_{Q'}) \end{matrix} \right. \right] \tag{2.1} $$

Where R is defined as

$$ R = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \neq 0, \tag{2.2} $$

$$ Re\,(h_i) > 0, Re\,(p_i) > 0 \;\; (i = 1,2), $$

$$ Re\,(h_1 + p_1 \frac{f_j}{F_j}) > 0 \qquad (1 \le 1 \le M'), $$

m is an arbitrary positive integer and the coefficients $A_{n,s}$ (n,s ≥ 0) are arbitrary constants, real or complex. The H-function occurring in (2.1) satisfies the conditions corresponding appropriately to those given in [7] and ρ' is neither zero nor a negative integer and Re (ρ' - b' - a') > 0.

Second Integral Transformation:

$$ \int_0^{\infty}\int_0^{\infty} (\alpha_1 y_1 + \alpha_2 y_2)^{h_1-1}(\beta_1 y_1 + \beta_2 y_2)^{h_2-1} $$

$$.exp\,[-p_1(\alpha_1 y_1+\alpha_2 y_2)-p_2(\beta_1 y_1+\beta_2 y_2)]$$

$$_2F_1\left({a',b' \atop \rho'};x_3(\alpha_1 y_1+\alpha_2 y_2\right)S_n^m(\alpha_1,y_1+\alpha_2,y_2)$$

$$.S_v^{\lambda,\mu,0}(\beta_1 y_1+\beta_2 y_2)\,H^{0,N:(M_1,N_1);(M_2,N_2)}_{P,Q:(P_1,Q_1);(P_2,Q_2)}\left[\begin{array}{c|c} x_1 & (\alpha_1 y_1+\alpha_2 y_2)^{\sigma_1} \\ x_2 & (\beta_1 y_1+\beta_2 y_2)^{\sigma_2}\end{array}\right]$$

$$.H^{M',N'}_{P',Q'}\left[(\alpha_1 y_1+\alpha_2 y_2)^{\sigma}\,\middle|\,\begin{array}{c}(e_{P'},E_{P'})\\(f_{Q'},F_{Q'})\end{array}\right]dy_1\,dy_2$$

$$=\frac{1}{R}\sum_{s=0}^{[n/m]}\frac{(-n)_{ms}}{s!}A_{n,s}\sum_{G,s'=0}^{\infty}\sum_{g=1}^{M'}\frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}\sum_{\ell=0}^{U+V}\sum_{k=0}^{\ell}$$

$$.\frac{(-\ell)_k\,v^{U+V}\,\mu^{\ell}\,(a')_{s'}(b')_{s'}\,x_3^{s'}}{s'!\,(\rho')_{s'}}\left(\frac{\lambda+\omega V+u+tk}{v}\right)_{U+V}$$

$$.\frac{1}{p_1^{(h_1+s+s'+\sigma\eta_G)}}\frac{1}{p_2^{\{h_2+\omega V+v(U+V)+t\ell\}}}H^{0,N:(M_1,N_1+1);(M_2,N_2+1)}_{P,Q:(P_1+1,Q_1);(P_2+1,Q_2)}\left[\begin{array}{c} x_1 p_1^{-\sigma_1} \\ x_2 p_2^{-\sigma_2}\end{array}\right|$$

$$\left.\begin{array}{l}(a_j:\alpha_j',\alpha_j'')_{1,P}:(1-h_1-s-s'-\eta_G\sigma,\sigma_1),(c_j',\gamma_j')_{1,P_1};(c_j'',\gamma_j'')_{1,P_2}\\(b_j:\beta_j',\beta_j'')_{1,Q}:(1-h_2-\omega V-v(U+V)-t\ell,\sigma_2),(d_j',\delta_j')_{1,Q_1};(d_j'',\delta_j'')_{1,Q_2}\end{array}\right], \quad (2.3)$$

Where R is defined in (2.1) and the following conditions are satisfied

(i) $\sigma_i \geq 0$, Re $(p_i) \geq 0$, $(i = 1,2)$

(ii) $Re(h_i+\sigma_i \frac{d_j}{\delta_j})>0 (i=1,2)$, $Re(\rho'-b'-a')>0$,

(iii) ρ' is neither zero nor a negative integer and Re (ρ' - b' - a')> 0,

(iv) m is an arbitrary positive integer and the coefficients $A_{n,s}$ $(n,s \geq 0)$ are arbitrary constants, real or complex. The H-function occurring in (2.3) satisfies the conditions corresponding appropriately to those given in [11].

Proof. To prove (2.1), we make use of the following known integral [12]

$$\int_0^{\infty}\int_0^{\infty}F(\alpha_1 y_1+\alpha_2 y_2,\beta_1,y_1+\beta_2 y_2)dy_1\,dy_2$$

$$=\frac{1}{R}\int_0^{\infty}\int_0^{\infty}F(u_1,u_2)du_1,du_2, \quad (2.4)$$

Where R is defined in (2.2).

If we take

$$F(\alpha_1 y_1+\alpha_2 y_2,\beta_1 y_1+\beta_2 y_2)$$
$$=f_1(\alpha_1 y_1+\alpha_2 y_2)f_1(\beta_1 y_1+\beta_2 y_2)$$

then

$$\int_0^\infty\int_0^\infty f_1(\alpha_1 y_1 + \alpha_2 y_2), f_2(\beta_1, y_1 + \beta_2 y_2) dy_1\, dy_2$$

$$= \frac{1}{R}\int_0^\infty f_1(u_1), du_1 \int_0^\infty f_2(u_2), du_2, \tag{2.5}$$

Consider

$$f_1(\alpha_1 y_1 + \alpha_2 y_2) = (\alpha_1 y_1 + \alpha_2 y_2)^{h_1 - 1}\exp[-p(\alpha_1 y_1 + \alpha_2 y_2)]$$

$$\cdot {}_2F_1\left(\begin{matrix} a', b' \\ \rho' \end{matrix}; x_3(\alpha_1 y_1 + \alpha_2 y_2\right) S_n^m(\alpha_1 y_1 + \alpha_2 y_2)$$

$$.H_{P',Q'}^{M',N'}[(\alpha_1 y_1 + \alpha_2 y_2)^\sigma]$$

and

$$f_2(\alpha_1 y_1 + \alpha_2 y_2) = (\beta_1 y_1 + \beta_2 y_2)^{h_2 - 1}\exp[-p_2(\beta_1 y_1 + \beta_2 y_2) S_V^{\lambda,\mu 0}(\beta_1 y_1 + \beta_2 y_2)$$

From (2.5), we find

$$\int_0^\infty\int_0^\infty f_1(\alpha_1 y_1 + \alpha_2 y_2)^{h_1 - 1}(\beta_1, y_1 + \beta_2 y_2)^{h_2 - 1}$$

$$\exp[-p_1(\alpha_1 y_1 + \alpha_2 y_2) - p_2(\beta_1 y_1 + \beta_2 y_2)]$$

$$\cdot {}_2F_1\left(\begin{matrix} a', b' \\ \rho' \end{matrix}; x_3(\alpha_1 y_1 + \alpha_2 y_2\right) S_n^m(\alpha_1 y_1 + \alpha_2 y_2)$$

$$.S_v^{?,\mu,0}(\beta_1 y_1 + \beta_2 y_2) H_{P',Q'}^{M',N'}[(\alpha_1 y_1 + \alpha_2 y_2)^\sigma] dy_1 dy_2$$

$$= \frac{1}{R}\int_0^\infty u_1^{h_2 - 1}\, ep^{-_1 u_1}\, {}_2F_1(u_1) S_n^m(u_1) H_{P',Q'}^{M',N'}[u_1^\sigma] du_1,$$

$$.\int_0^\infty u_2^{h_2 - 1}\, e^{-p_2 u_2} S_v^{?,\mu,0}(u_2) du_2,$$

On expressing all polynomials, Gauss function and Fox's H-function in series form and interchanging the order of integrals and summations and evaluating the u_1 and u_2 integrals with the help of a known result [3], we arrive at the desired result.
The result in (2.3) can be proved in the similar manner.

Special Cases:

- Taking $\alpha_1 = 1\ \beta_2, \alpha_2 = 0 = \beta_2$ in (2.3) and reduce the polynomial $S_n^m(y_1)$ in terms of Gould and Hopper polynomials $g_n^m(y,h)$ [3] and the generalized polynomials set $S_v^{?,\mu,0}(y_2)$ in terms of Gould and Hopper polynomials $H_V^1(y_2, ?, \mu)$ [3], we arrive at the following bi Laplace transform:

$$L\left\{y_1^{h_1 - 1} y_2^{h_2 - 1} g_n^m(y_1, h)\, H_V^t(y_2, \lambda, \mu)\ {}_2F_1\left(\begin{matrix} a', b' \\ \rho' \end{matrix}; x_3 y_1\right)\right.$$

$$.H(x_1 y_1^{\sigma_1}, x_2 y_2^{\sigma_2}) H_{P',Q'}^{M',N'}(y_1^{\sigma_1}) : p_1, p_2 \Big\}$$

$$= \frac{1}{R} \sum_{s=0}^{[n/m]} \sum_{G,s'=0}^{\infty} \sum_{g=1}^{M'} \sum_{\ell=0}^{V} \sum_{k=0}^{\ell} \frac{(ms)! \binom{n}{ms} h^s x_3^{s'} (-\ell)_k (-\lambda - kt)_v \mu^\ell (-1)^\ell}{s!\ s'!\ k!\ \ell!\ p_1^{h_1'} p_2^{h_2 - v + \ell t}}$$

$$H_{P,Q:(P_1+1,Q_1);(P_2+1,Q_2)}^{0,N:(M_1,N_1+1);(M_2,N_2+1)} \left[\begin{matrix} x_1 p_1^{-\sigma_1} \\ x_2 p_2^{-\sigma_2} \end{matrix} \right|$$

$$\left. \begin{matrix} (a_j:\alpha_j',\alpha_j'')_{1,P} : (1-h_1-\sigma_1), (c_j',\gamma_j')_{1,P_1} \ ; \ (1-h_2-V-\ell t, \sigma_2), (c_j'',\gamma_j'')_{1,P_2} \\ (b_j:\beta_j',\beta_j'')_{1,Q} : \ (d_j',\delta_j')_{1,Q_1} \ ; \ (d_j'',\delta_j'')_{1,Q_2} \end{matrix} \right], \tag{3.1}$$

Where $h_1' = h_1 + \sigma\eta_G + (n - ms - s')$ and the conditions easily obtainable from those stated with (2.3) are satisfies.

- Taking $x_3 = 0 = \sigma = \alpha_2 = \beta_1$ and $\alpha_1 = \beta_1 = 1$ in (2.3) and reduce the H-function of two variables in terms of product of a Whittaker and a modified Bessel's function [7], the polynomials $S_n^m(y_1)$ in terms of Hermite polynomials $H_x\left[\frac{1}{\sqrt{y_1}}\right]$ with the help of [14] and the polynomial set $S_v^{\lambda,\mu,0}(y_2)$ in terms of Laguerre polynomials $L_v^{(\lambda)}(y_2)$ by taking $\beta = \gamma = 1$ in [9], we arrive at a result given in Gupta and Agrawal [15, 16].
- The results obtained [14] follow as special cases of our results.

REFERENCES

1. Rainville, E.D., 1967. Special Functions, New York. The MacMillan Co.
2. Fox, C., 1961. The G and H-functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc., 98: 396-429.
3. Srivastava, H.M., 1972. A contour integral involving Fox's H-function, Indian J. Math., 14: 1-6.
4. Raizada, S.K., 1991. A study of Unified Representation of Special Functions of Mathematical Physics and Their Use in Statistical and Boundary Value Problems, Ph.D. Thesis, Bundelkhand Univ., India.
5. Mittal, P.K. and K.C. Gupta, 1972. An integral involving generalized function of two variables, Proc. Indian Acad. Sci. Sect. A, 75: 117-125.
6. Skibinski, P., 1970. Some expansion theorems for the H-function. Ann. Polon. Math., 23: 125-138.
7. Srivastava, H.M., K.C. Gupta and S.P. Goyal, 1982. The H-Functions of One and Two Variables with Applications, New Delhi and Madras, South Asian Publishers.
8. Krall, H.L. and O. Frink, 1949. A new class of polynomials: The Bessel polynomials, Trans. Amer. Math. Soc., 65: 100-115.
9. Tibor K. Pogány, 2007. Applied Mathematics Letters, 20(7): 764-769.
10. Tibor K. Pogány and Živorad Tomovski, 2008. Mathematical and Computer Modelling, 47(9-10): 952-969.
11. Gradshteyn, I.S. and I.M. Ryzhik, 1980. Tables of Integrals, Series and Products, New York. Academic Press, Inc.
12. Widder, D.V., 1968. Advanced Calculus, New York. Dover Publications Inc.
13. Gould, H.W. and A.I. Hopper, 1962. Operational formulas connected with two generalizations of Hermite Polynomials, Duke Math. J., 29: 51-63 1962.
14. Srivastava, H.M. and N.P. Singh, 1983. The integration of certain products of the multivariable H-function with a general class of polynomials, Rend. Circ. Mat. Palermo., 32(2): 157-187.
15. Garg, Mridula, and Gupta, Mukesh Kumar, 1997. Some double integrals and Laplace transforms associated with them, Jñ ā n ā bha, 27: 123-130.
16. Gupta, K.C. and Agarwal, Pawan 1995. A study of unified double and multivariable integrals and the Laplace transforms associated with them, Bull. Cal. Mat. Soc., 87: 299-306.