

## A Study on Systems of Variable-Coefficient Singular Parabolic Partial Differential Equations

<sup>1</sup>Syed Tauseef Mohyud-Din, <sup>2</sup>Ahmet Yildirim,  
<sup>3</sup>M.M. Hosseini and <sup>4</sup>Y. Khan

<sup>1</sup>Department of Mathematics, HITEC University Taxila Cantt Pakistan

<sup>2</sup>Ege University, Science Faculty, Department of Mathematics, 35100 Bornova- Izmir, Turkey

<sup>3</sup>Faculty of Mathematics, Yazd University, P. O. Box 89195-74, Yazd, Iran.

<sup>4</sup>Modern Textile Institute, Donghua University, 1882 Yan'an Xilu Road, Shanghai 200051, China

---

**Abstract:** This paper reflects the implementation of homotopy perturbation method (HPM) on the re-formulated systems of fourth-order parabolic partial differential equations. Numerical results explicitly reveal the complete reliability of the proposed algorithm.

**Key words:** Homotopy perturbation method • Singular fourth-order parabolic PDES • Boundary value problems

### INTRODUCTION

This paper is devoted to the study of the singular fourth-order parabolic partial differential equations [1-7] with variable coefficient. It is well known in the literature that a wide class of problems arising in mathematics, physics, astrophysics and engineering sciences can be distinctively formulated as singular initial and boundary value problems. The singular fourth-order parabolic partial differential equations govern the transverse vibrations of a homogeneous beam. Such types of equations arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory, see [6, 7]. Several techniques [1-7] including variational iteration, decomposition, explicit and implicit finite difference schemes, lines approach and separation of variables have been applied to tackle such problems. Inspired and motivated by the ongoing research in this area, we apply homotopy perturbation method (HPM) to solve the singular fourth-order parabolic partial differential equations with variable coefficient. The singular fourth-order parabolic partial differential equations are converted to a system of partial differential equations by introducing a suitable transformation. The proposed HPM is applied to the resultant system of integro partial differential equations.

Several examples are given to verify the reliability and efficiency of the proposed algorithm.

**Homotopy Perturbation Method (HPM):** To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(u)=0, \quad (1)$$

Where L is any integral or differential operator. We define a convex homotopy  $H(u, p)$  by

$$H(u, p) = (1-p)F(u) + pL(u), \quad (2)$$

Where  $F(u)$  is a functional operator with known solutions  $v_0$ , which can be obtained easily. It is clear that, for

$$H(u, p) = 0 \quad (3)$$

We have

$$H(u, p) = F(u), \quad H(u, 1) = L(u)$$

This shows that  $H(u, p)$  continuously traces an implicitly defined curve from a starting point  $H(v_0, 0)$  to a solution function  $H(f, 1)$ . The embedding parameter monotonically increases from zero to unit as the trivial problem  $F(u) = 0$  is continuously deforms the original

problem  $L(u) = 0$ . The embedding parameter  $p \in (0, 1]$  can be considered as an expanding parameter [8-23]. The homotopy perturbation method (HPM) uses the homotopy parameter  $p$  as an expanding parameter [8-23] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots, \quad (4)$$

If  $p \rightarrow 1$ , then (4) corresponds to (2) and becomes the approximate solution of the form,

$$f = \lim_{p \rightarrow 1} u = \sum_{i=0}^{\infty} u_i. \quad (5)$$

It is well known that series (5) is convergent for most of the cases and also the rate of convergence is dependent on  $L(u)$ ; see [8-23]. We assume that (5) has a unique solution. The comparisons of like powers of  $p$  give solutions of various orders.

**Numerical Applications:** In this section, we apply HPM to solve the re-formulated fourth-order parabolic partial differential equations. Several examples are given to verify the reliability and efficiency of the proposed algorithm.

**Example 3.1:** Consider the following fourth-order singular parabolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 u}{\partial x^4} = 0, \quad 0 < x < 1, \quad t > 0$$

With initial conditions

$$\begin{aligned} u(x, 0) &= x - \sin x, \quad 0 < x < 1, \\ \frac{\partial u}{\partial t}(x, 0) &= -(x - \sin x), \quad 0 < x < 1 \end{aligned}$$

and the boundary conditions

$$\begin{aligned} u(0, t) &= 0, \quad u(1, t) = e^{-t}(1 - \sin 1), \quad t > 0, \\ \frac{\partial^2 u}{\partial x^2}(0, t) &= 0, \quad \frac{\partial^2 u}{\partial x^2}(1, t) = e^{-t} \sin 1, \quad t > 0. \end{aligned}$$

Using the transformation  $\frac{\partial u}{\partial t} = q(t)$ , the above

problem can be converted to the following system of partial differential equations

$$\begin{cases} \frac{\partial u}{\partial t} = q(t), \\ \frac{\partial q}{\partial t} = -\left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 u}{\partial x^4}, \end{cases}$$

With initial conditions

$$u(x, 0) = x - \sin x, \quad q(x, 0) = -(x - \sin x).$$

The above system of differential equations can be converted to the following system of integral equations

$$\begin{cases} u(x, t) = (x - \sin x) + \int_0^t q(t) dt, \\ q(x, t) = -(x - \sin x) + \int_0^t \left( -\frac{x}{\sin x} - 1 \right) \frac{\partial^4 u}{\partial x^4} dt, \end{cases}$$

Applying homotopy perturbation method (HPM), we get

$$\begin{cases} u_0 + p u_1 + p^2 u_2 + \dots = (x - \sin x) + \\ p \int_0^t \left( (q_0 + p q_1 + p^2 q_2 + \dots) \right) dt, \\ q_0 + p q_1 + p^2 q_2 + \dots = -(x - \sin x) + \\ p \int_0^t \left( \left( -\frac{x}{\sin x} - 1 \right) \left( \frac{\partial^4 u_0}{\partial x^4} + p \frac{\partial^4 u_1}{\partial x^4} + \dots \right) \right) dt. \end{cases}$$

Comparing the co-efficient of like powers of  $p$ , following approximants are obtained

$$\begin{aligned} p^{(0)}: & \begin{cases} u_0(x, t) = x - \sin x, \\ q_0(x, t) = -(x - \sin x), \end{cases} \\ p^{(1)}: & \begin{cases} u_1(x, t) = x - \sin x - (x - \sin x)t, \\ q_1(x, t) = -(x - \sin x) + (x - \sin x) \left( t - \frac{t^2}{2!} \right) \end{cases} \\ p^{(2)}: & \begin{cases} u_2(x, t) = (x - \sin x) - (x - \sin x)t + (x - \sin x) \left( \frac{t^2}{2!} - \frac{t^3}{3!} \right), \\ q_2(x, t) = -(x - \sin x) + (x - \sin x) \left( t - \frac{t^2}{2!} \right) + (x - \sin x) \left( \frac{t^3}{3!} - \frac{t^4}{4!} \right), \end{cases} \end{aligned}$$

$$u(x, y, 0) = 0, \quad \frac{\partial u}{\partial t}(x, y, 0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!},$$

$$\begin{aligned}
 p^{(3)} : & \begin{cases} u_3(x,t) = (x - \sin x) - (x - \sin x)t + (x - \sin x) \\ \left( \frac{t^2}{2!} - \frac{t^3}{3!} \right) + (x - \sin x) \left( \frac{t^4}{4!} - \frac{t^5}{5!} \right), \\ q_3(x,t) = -(x - \sin x) + (x - \sin x) \left( t - \frac{t^2}{2!} \right) + \\ (x - \sin x) \left( \frac{t^3}{3!} - \frac{t^4}{4!} \right) + (x - \sin x) \left( \frac{t^5}{5!} - \frac{t^6}{6!} \right), \end{cases} \\
 & u_4(x,t) = (x - \sin x) - (x - \sin x)t + (x - \sin x) \\ & \left( \frac{t^2}{2!} - \frac{t^3}{3!} \right) + (x - \sin x) \left( \frac{t^4}{4!} - \frac{t^5}{5!} \right) + (x - \sin x) \left( \frac{t^6}{6!} - \frac{t^7}{7!} \right), \\
 p^{(4)} : & q_4(x,t) = -(x - \sin x) + (x - \sin x) \left( t - \frac{t^2}{2!} \right) + (x - \sin x) \\ & \left( \frac{t^3}{3!} - \frac{t^4}{4!} \right) + (x - \sin x) \left( \frac{t^5}{5!} - \frac{t^6}{6!} \right) + (x - \sin x) \left( \frac{t^7}{7!} - \frac{t^8}{8!} \right), \\
 & u_5(x,t) = (x - \sin x) - (x - \sin x)t + (x - \sin x) \left( \frac{t^2}{2!} - \frac{t^3}{3!} \right) \\ & + (x - \sin x) \left( \frac{t^4}{4!} - \frac{t^5}{5!} \right) + (x - \sin x) \left( \frac{t^6}{6!} - \frac{t^7}{7!} \right) + (x - \sin x) \left( \frac{t^8}{8!} - \frac{t^9}{9!} \right), \\
 p^{(5)} : & q_5(x,t) = -(x - \sin x) + (x - \sin x) \left( t - \frac{t^2}{2!} \right) + (x - \sin x) \left( \frac{t^3}{3!} - \frac{t^4}{4!} \right) \\ & + (x - \sin x) \left( \frac{t^5}{5!} - \frac{t^6}{6!} \right) + (x - \sin x) \left( \frac{t^7}{7!} - \frac{t^8}{8!} \right) + (x - \sin x) \left( \frac{t^9}{9!} - \frac{t^{10}}{10!} \right).
 \end{cases}
 \end{aligned}$$

The solution is given as

$$u(x,t) = (x - \sin x) \left( 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \dots \right) = (x - \sin x)e^{-t}.$$

**Example 3.2:** Consider the following singular fourth order parabolic partial differential equation in two space variables

$$\frac{\partial^2 u}{\partial t^2} + 2 \left( \frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 u}{\partial x^4} + 2 \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 u}{\partial y^4} = 0,$$

With initial conditions

$$u(x,y,0) = 0, \quad \frac{\partial u}{\partial t}(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!},$$

and the boundary conditions

$$\begin{aligned}
 u\left(\frac{1}{2}, y, t\right) &= \left( 2 + \frac{(0.5)^6}{6!} + \frac{y^6}{6!} \right) \sin t, \quad u(1, y, t) = \left( 2 + \frac{1}{6!} + \frac{y^6}{6!} \right) \sin t, \\
 \frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, y, t\right) &= \frac{(0.5)^4}{24} \sin t, \quad \frac{\partial^2 u}{\partial x^2}(1, y, t) = \frac{1}{24} \sin t, \\
 \frac{\partial^2 u}{\partial y^2}\left(x, \frac{1}{2}, t\right) &= \frac{(0.5)^4}{24} \sin t, \quad \frac{\partial^2 u}{\partial y^2}(x, 1, t) = \frac{1}{24} \sin t.
 \end{aligned}$$

Using the transformation  $\frac{\partial u}{\partial t} = q(t)$ , the above problem can be converted to the following system of partial differential equations

$$\begin{cases} \frac{\partial u}{\partial t} = q(t), \frac{\partial q}{\partial t} = -2 \left( \frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 u}{\partial x^4} \\ -2 \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 u}{\partial y^4}, \end{cases}$$

With initial conditions

$$u(x, y, 0) = 0, \quad q(x, y, 0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}$$

The above system of differential equations can be converted to the following system of integral equations

$$\begin{cases} u(x, t) = \int_0^t q(t) dt, \quad q(x, t) = - \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \\ -2 \left( \int_0^t \left( \frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 u}{\partial x^4} + \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 u}{\partial y^4} \right) dt. \end{cases}$$

Applying homotopy perturbation method (HPM), we get

$$\begin{cases} u_0 + pu_1 + p^2 u_2 + \dots = p \int_0^t \left( (q_0 + pq_1 + p^2 q_2 + \dots) \right) dt, \\ q_0 + pq_1 + p^2 q_2 + \dots = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) - p \int_0^t \left( \begin{array}{l} \left( \frac{2}{x^2} + \frac{x^4}{6!} \right) \\ \left( \frac{\partial^4 u_0}{\partial x^4} + p \frac{\partial^4 u_1}{\partial x^4} \right) \\ + p^2 \frac{\partial^4 u_2}{\partial x^4} + \dots \end{array} \right) dt \\ -2p \int_0^t \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \left( \frac{\partial^4 u_0}{\partial y^4} + p \frac{\partial^4 u_1}{\partial y^4} + p^2 \frac{\partial^4 u_2}{\partial y^4} + \dots \right) dt. \end{cases}$$

Comparing the co-efficient of like powers of p, following approximants are obtained

$$\begin{aligned}
 p^{(0)} : & \begin{cases} u_0(x,t) = 0, \\ q_0(x,t) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}, \end{cases} \\
 p^{(1)} : & \begin{cases} u_1(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) t, \\ q_1(x,t) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}, \end{cases} \\
 p^{(2)} : & \begin{cases} u_2(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) t, \\ q_2(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( 1 - \frac{t^2}{2!} \right), \end{cases} \\
 p^{(3)} : & \begin{cases} u_3(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( t - \frac{t^3}{3!} \right), \\ q_3(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \right), \end{cases} \\
 p^{(4)} : & \begin{cases} u_4(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} \right), \\ q_4(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( 1 - \frac{t^2}{2!} + \frac{t^4}{5!} - \frac{t^5}{6!} \right), \end{cases} \\
 p^{(5)} : & \begin{cases} u_5(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \right), \\ q_5(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} \right), \end{cases} \\
 p^{(6)} : & \begin{cases} u_6(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} \right), \\ q_6(x,t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \frac{t^{10}}{10!} \right), \end{cases} \\
 \vdots &
 \end{aligned}$$

The exact solution is recognized easily

$$\begin{aligned}
 u(x,y,t) &= \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} + \dots \right) \\
 &= \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \sin t.
 \end{aligned}$$

**Example 3.3:** Consider the following three dimensional non-homogeneous singular parabolic partial differential equation

$$\begin{aligned}
 & \frac{\partial^2 u}{\partial t^2} + \left( \frac{1}{4!z} \right) \frac{\partial^4 u}{\partial x^4} + \left( \frac{1}{4!x} \right) \frac{\partial^4 u}{\partial y^4} + \left( \frac{1}{4!y} \right) \frac{\partial^4 u}{\partial z^4} \\
 &= - \left[ \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right] \cos t,
 \end{aligned}$$

With initial conditions

$$u(x,y,z,0) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}, \quad \frac{\partial u}{\partial t}(x,y,z,0) = 0,$$

and the boundary conditions

$$\begin{aligned}
 u\left(\frac{1}{2}, y, z, t\right) &= \left( \frac{1}{2y} + \frac{y}{z} + 2z \right) \cos t, \quad u(1, y, z, t) = \left( \frac{1}{y} + \frac{y}{z} + z \right) \cos t, \\
 u\left(x, \frac{1}{2}, z, t\right) &= \left( 2x + \frac{1}{2z} + \frac{z}{x} \right) \cos t, \quad u(x, 1, z, t) = \left( x + \frac{1}{z} + \frac{z}{x} \right) \cos t, \\
 u\left(x, y, \frac{1}{2}, t\right) &= \left( 2y + \frac{x}{y} + \frac{1}{2x} \right) \cos t, \quad u(x, y, 1, t) = \left( y + \frac{x}{y} + \frac{1}{x} \right) \cos t,
 \end{aligned}$$

Using the transformation  $\frac{\partial u}{\partial t} = q(t)$ , the above problem can be converted to the following system of partial differential equations

$$\begin{cases} \frac{\partial u}{\partial t} = q(t), \\ \frac{\partial q}{\partial t} = - \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 u}{\partial x^4}, \end{cases}$$

With initial conditions

$$u(x,0) = x - \sin x, \quad q(x,0) = -(x - \sin x).$$

The above system of differential equations can be converted to the following system of integral equations

$$\begin{cases} u(x,t) = (x - \sin x) + \int_0^t q(t) dt, \\ q(x,t) = -(x - \sin x) + \int_0^t \left( -\frac{x}{\sin x} - 1 \right) \frac{\partial^4 u}{\partial x^4} dt, \end{cases}$$

Applying homotopy perturbation method (HPM), we get

$$\begin{cases} u_0 + pu_1 + p^2 u_2 + \dots = (x - \sin x) \\ + p \int_0^t \left( (q_0 + pq_1 + p^2 q_2 + \dots) \right) dt, \\ q_0 + pq_1 + p^2 q_2 + \dots = -(x - \sin x) \\ + p \int_0^t \left( \left( -\frac{x}{\sin x} - 1 \right) \left( \frac{\partial^4 u_0}{\partial x^4} + p \frac{\partial^4 u_1}{\partial x^4} + \dots \right) \right) dt. \end{cases}$$

Comparing the co-efficient of like powers of  $p$ , following approximants are obtained

$$p^{(0)} : \begin{cases} u_0(x, t) = x - \sin x, \\ q_0(x, t) = -(x - \sin x), \end{cases}$$

$$p^{(1)} : \begin{cases} u_1(x, t) = x - \sin x - (x - \sin x)t, \\ q_1(x, t) = -(x - \sin x) + (x - \sin x) \left( t - \frac{t^2}{2!} \right) \end{cases}$$

$$p^{(2)} : \begin{cases} u_2(x, t) = (x - \sin x) - (x - \sin x)t + (x - \sin x) \left( \frac{t^2}{2!} - \frac{t^3}{3!} \right), \\ q_2(x, t) = -(x - \sin x) + (x - \sin x) \left( t - \frac{t^2}{2!} \right) + (x - \sin x) \left( \frac{t^3}{3!} - \frac{t^4}{4!} \right), \\ u_3(x, t) = (x - \sin x) - (x - \sin x)t + (x - \sin x) \left( \frac{t^2}{2!} - \frac{t^3}{3!} \right) + (x - \sin x) \left( \frac{t^4}{4!} - \frac{t^5}{5!} \right), \end{cases}$$

$$p^{(3)} : \begin{cases} q_3(x, t) = -(x - \sin x) + (x - \sin x) \left( t - \frac{t^2}{2!} \right) + (x - \sin x) \left( \frac{t^3}{3!} - \frac{t^4}{4!} \right) + (x - \sin x) \left( \frac{t^5}{5!} - \frac{t^6}{6!} \right), \end{cases}$$

The sequences tends to  $\left( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \cos t$ , as  $n \rightarrow \infty$ ,

therefore, the exact solution is given as

$$u(x, y, z, t) = \left( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \cos t.$$

**Example 3.4 Consider the Following Singular Fourth-order Parabolic Equation:**

$$\frac{\partial^2 u}{\partial t^2} + \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 u}{\partial x^4} = 0, \quad \frac{1}{2} < x < 1, t > 0$$

subject to the initial conditions

$$\begin{aligned} u(x, 0) &= 0, & \frac{1}{2} < x < 1, \\ \frac{\partial u}{\partial t}(x, 0) &= 1 + \frac{x^5}{120}, & \frac{1}{2} < x < 1 \end{aligned}$$

and the boundary conditions

$$\begin{aligned} u\left(\frac{1}{2}, t\right) &= \left( 1 + \frac{(1/2)^5}{120} \right) \sin t, & u(1, t) &= \left( \frac{121}{120} \right) \sin t, \quad t > 0, \\ \frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, t\right) &= \frac{1}{6} \left( \frac{1}{2} \right)^3 \sin t, & \frac{\partial^2 u}{\partial x^2}(1, t) &= \frac{1}{6} \sin t, \quad t > 0. \end{aligned}$$

Using the transformation  $\frac{\partial u}{\partial t} = q(t)$ , the above problem can be converted to the following system of partial differential equations

$$\begin{cases} \frac{\partial u}{\partial t} = q(t), \\ \frac{\partial q}{\partial t} = -\left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 u}{\partial x^4}, \end{cases}$$

with initial conditions

$$u(x, 0) = 0, \quad q(x, 0) = 1 + \frac{x^5}{120}.$$

Applying homotopy perturbation method (HPM), we get

$$\begin{cases} u_0 + pu_1 + p^2 u_2 + \dots = p \int_0^t \left( q_0 + pq_1 + p^2 q_2 + \dots \right) dt, \\ q_0 + pq_1 + p^2 q_2 + \dots = \left( 1 + \frac{x^5}{120} \right) - p \int_0^t \left( \frac{1}{x} + \frac{x^4}{120} \right) \left( \frac{\partial^4 u_0}{\partial x^4} + p \frac{\partial^4 u_1}{\partial x^4} + p^2 \frac{\partial^4 u_2}{\partial x^4} + \dots \right) dt. \end{cases}$$

Comparing the co-efficient of like powers of  $p$ , following approximants are obtained

$$p^{(0)} : \begin{cases} u_0(x, t) = 0, \\ q_0(x, t) = 1 + \frac{x^5}{120}, \end{cases}$$

$$p^{(1)} : \begin{cases} u_1(x, t) = \left( 1 + \frac{x^5}{120} \right) t, \\ q_1(x, t) = 1 + \frac{x^5}{120}, \end{cases}$$

$$\begin{aligned}
 p^{(2)} : & \begin{cases} u_2(x,t) = \left(1 + \frac{x^5}{120}\right)t, \\ q_2(x,t) = \left(1 + \frac{x^5}{120}\right) - \left(1 + \frac{x^5}{120}\right)\frac{t^2}{2!}, \end{cases} \\
 p^{(3)} : & \begin{cases} u_3(x,t) = \left(1 + \frac{x^5}{120}\right)t - \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!}, \\ q_3(x,t) = \left(1 + \frac{x^5}{120}\right) - \left(1 + \frac{x^5}{120}\right)\frac{t^2}{2!}, \end{cases} \\
 p^{(4)} : & \begin{cases} u_4(x,t) = \left(1 + \frac{x^5}{120}\right)t - \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!}, \\ q_4(x,t) = \left(1 + \frac{x^5}{120}\right) - \left(1 + \frac{x^5}{120}\right)\frac{t^2}{2!} - \left(1 + \frac{x^5}{120}\right)\frac{t^4}{4!}, \end{cases} \\
 \vdots &
 \end{aligned}$$

The solution in a series form is

$$u(x,t) = \left(1 + \frac{x^5}{120}\right) \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \right) = \left(1 + \frac{x^5}{120}\right) \sin t.$$

## CONCLUSION

In this paper, we applied the homotopy perturbation method (HPM) to solve re-formulated systems of singular fourth-order parabolic partial differential equations with variable co-efficient. The fact that the proposed algorithm solves nonlinear problems without using the Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

## REFERENCE

1. Abbasbandy, S., 2007. Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method, *Internat. J. Numer. Meth. Engrg.*, 70: 876-881.
2. Abbasbandy, S., 2008. Numerical method for nonlinear wave and diffusion equations equation by the variational iteration method, *Internat. J. Numer. Meth. Engrg.*, 73: 1836-1843.
3. Abbasbandy, S. and A. Shirzadi, 2008. The variational iteration method for a class of eighth-order boundary value differential equations, *Zeitschrift für Naturforschung. A.*, 63a: 745-751.
4. Abdou, M.A. and A.A. Soliman, 2005. Variational iteration method for solving Burger's and coupled Burger's equations, *J. Comput. Appl. Math.*, 181: 245-251.
5. Abdou, M.A. and A.A. Soliman, 2005. New applications of variational iteration method, *Phys. D.*, 211(1-2): 1-8.
6. Mohyud-Din S.T. and A. Yıldırım, 2010. Latest developments in nonlinear sciences, *Applications and Applied Mathematics, Special Issue*, 1 : 46-72.
7. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Some relatively new techniques for nonlinear problems, *Mathematical Problems in Engineering*, Hindawi, 2009 (2009); Article ID 234849, 25 pages, doi:10.1155/2009/234849.
8. Ghorbani A. and J.S. Nadjfi, 2007. He's homotopy perturbation method for calculating Adomian's polynomials, *Int. J. Nonlin. Sci. Num. Simul.*, 8(2): 229-332.
9. He, J.H., 2006. Some asymptotic methods for strongly nonlinear equation, *Int. J. Mod. Phy.*, 20(20)10: 1144-1199.
10. He, J.H., 1999. Homotopy perturbation technique, *Comput. Math. Appl. Mech. Engg.*, pp: 178-257.
11. He, J.H., 2006. Homotopy perturbation method for solving boundary value problems, *Phy. Lett. A.*, 350: 87-88.
12. He, J.H., 2004. Comparison of homotopy perturbation method and homotopy analysis method, *Appl. Math. Comput.*, 156: 527-539.
13. He, J.H., 2005. Homotopy perturbation method for bifurcation of nonlinear problems, *Int. J. Nonlin. Sci. Numer. Simul.*, 6(2): 207-208.
14. He, J.H. ,2004. The homotopy perturbation method for nonlinear oscillators with discontinuities, *Appl. Math. Comput.*, 151: 287-292.
15. He, J.H., 2000. A coupling method of homotopy technique and perturbation technique for nonlinear problems, *Int. J. Nonlin. Mech.*, 35(1): 115-123.
16. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Travelling wave solutions of seventh-order generalized KdV equations using He's polynomials, *Int. J. Nonlin. Sci. Num. Sim.*, 10(2): 223-229.
17. Mohyud-Din S.T. and M.A. Noor, 2007. Homotopy perturbation method for solving fourth-order boundary value problems, *Math. Prob. Eng.* 2007 (2007), 1-15, ArticleID 98602, doi:10.1155/2007/98602.
18. Mohyud-Din S.T. and M.A. Noor, 2008. Homotopy perturbation method for solving partial differential equations, *Zeitschrift für Naturforschung. A.*, 64a: 1-14.

19. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Parameter-expansion techniques for strongly nonlinear oscillators, *International J. Nonlinear Sciences and Numerical Simulation*, 10(5): 581-583.
20. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Variational iteration method for Flierl-Petviashvili equations using He's polynomials and Pade'approximants, *World Applied Sciences Journal*, 6(9): 1298-1303.
21. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Solving second-order singular problems using He's polynomials, *World Applied Sciences J.*, 6(6): 769-775.
22. Noor, M.A. and S.T. Mohyud-Din, 2008. Homotopy perturbation method for nonlinear higher-order boundary value problems, *Int. J. Nonlin. Sci. Num. Simul.*, 9(4): 395-408.
23. Noor M.A. and S.T. Mohyud-Din, 2008. Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials, *International J. Nonlinear Sciences and Numerical Simulation*, 9(2): 141-157; IF = 8.479.
00. Xu, L., 2007. He's homotopy perturbation method for a boundary layer equation in unbounded domain, *Comput. Math. Appl.*, 54: 1067-1070.