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Parametric Iteration Method: A useful Analytic Method for Solving Riccati Differential Equations

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Abstract: In this paper, the Parametric Iteration Method (PIM) is first proposed for solving Riccati Differential Equations (RDEs). The original PIM provides the solution of a RDE as a sequence of approximations. A new application of the PIM is then given for handling RDEs, which provides the solution of a RDE as a series of approximations. The analyzed example reveals that the developed analytical algorithms are simple and effective to solve RDEs .

Key words: Parametric iteration method. Riccati differential equations

INTRODUCTION

The Riccati differential equations are a class of nonlinear differential equations of much importance, and it plays a significant role in many fields of applied science. For example, a well-known one-dimensional static Schrödinger equation is closely related to the RDE. Solitary wave solutions of a nonlinear partial differential equation can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation. In this work, we consider a RDE of the following form:

$$u'(t) = A(t) + B(t)u + C(t)u^2, \qquad u(t_0) = c \tag{1}$$

where A(t), B(t) and C(t) are given functions and c is an arbitrary constant. The importance of this equation usually arises in the optimal control problems . The feed back gain of the linear quadratic optimal control depends on a solution of a RDE which has to be found for the whole time horizon of the control process [1]. Now a days, deriving its analytical solution in an explicit form seems to be unlikely except for certain special situations. Of course, if one particular solution of Eq. (1) is known, then its general solution can easily be found. For general cases, one must appeal to numerical techniques or approximate approaches for getting its solutions.

Recently, due to its importance in many fields of applied sciences, a vast amount of research work has been invested in the study of numerical and analytical of Eq. (1). Recently, El-Tawil *et al*. [2] applied the multistage Adomian Decomposition Method (ADM) to solving the RDE and compared the results with the standard ADM. Tan and Abbasbandy in [3] employed the Homotopy Analysis Method (HAM) to solve this equation. Newly, Abbasbandy [4] applied the Homotopy Perturbation Method (HPM), which is a special case of the HAM, to solve the RDE and compared the obtained results for this equation. Also, Abbasbandy [5] used the iterated HPM to solve this equation, the obtained results were better for the long time horizon. Furthermore, in [6, 7], the Variational Iteration Method (VIM) was utilized for solving Eq. (1). More recently, Geng *et al*. proposed a piecewise VIM for the RDEs [8].

In this paper, the effective Parametric Iteration Method (PIM) [9] is presented for finding the approximate analytical solution of the RDE (1). It is shown that the PIM reasonable includes all the above-mentioned approximate method.

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PARAMETRIC ITERATION METHOD

In this section, the PIM and new applications of that are proposed for solving Eq. (1), which is capable of providing the solution both as a sequence and series.

THE PIM EXPRESSED BY SEQUENCE

In this subsection, we first describe the PIM expressed by sequence for solving the RDEs of (1), which is the main feature of the PIM. The PIM gives rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise approximations can be used for numerical purposes. The idea of the PIM expressed by sequence is very simple and straightforward.

To explain the basic idea of the PIM expressed by sequence, we first consider Eq. (1) as below:

$$L[u(t)]+N[u(t)]=g(t) \tag{2}$$

where L denotes a continuous auxiliary linear operator with respect to u, N is a continuous nonlinear operator with respect to u and g(t) is a given analytic function. Then, we construct a family of iterative processes for Eq. (2) as [9, 10]:

$$u_{n+1}(t)=u_n(t)-h\int_{t_0}^{t}\Lambda_{(t,s)}(H(s)\{L[u_n(s)]+N[u_n(s)]-g(s)\})ds \tag{3}$$

with the property

$$u_n(t_0)=u(t_0), \quad \forall n \tag{4}$$

where $u_0(t)$ is the initial guess, which can be freely chosen from solving its corresponding linear equation $L[u_0(t)]=0$ or $L[u_0(t)]=g(t)$ and the subscript n denotes the nth iteration. $\Lambda_{(t,s)}\neq 0$ denotes the general multiplier, which taking into account the auxiliary linear operator as $L[u]=u'-B(t)u$ can be identified easily and efficiently by solving the following linear conditions [10, 12]:

$$\frac{\partial}{\partial s}\Lambda_{(t,s)}+B(s)\Lambda_{(t,s)}=0, \ \Lambda_{(t,t)}=-1 \tag{5}$$

which gives us

$$\Lambda_{(t,s)}=-\exp\left(\int_s^t B(\xi)d\xi\right) \tag{6}$$

Using the relations (1), (3) and (6), we find that $u_n(t), n\geq 1$ can be obtained in general by

$$u_{n+1}(t)=u_n(t)+h\int_{t_0}^{t}\exp\left(\int_s^t B(\xi)d\xi\right)\times(H(s)\{u_n'(s)-A(s)-B(s)u_n(s)-C(s)u_n^2(s)\})ds \tag{7}$$

which is called the PIM expressed by sequence of first kind. Now let us consider (7) as below:

$$u_{n+1}(t)=u_n(t)+h\int_{t_0}^{t}\exp\left(\int_s^t B(\xi)d\xi\right)[u_n'(s)-B(s)u_n(s)]ds-h\int_{t_0}^{t}\exp\left(\int_s^t B(\xi)d\xi\right)H(s)[A(s)+C(s)u_n^2(s)]ds \tag{8}$$

By using simple integration by parts, we get

$$h\int_{t_0}^{t}\exp\left(\int_s^t B(\xi)d\xi\right)L[u_n(s)]ds=hu_n(t)-hu_0(t) \tag{9}$$

where $u_0(t)=u(t_0)\exp(\int_s^t B(\xi)d\xi)$. Here, in view of (1), (4) and (9), Eq. (8) can be rewritten as

$$u_{n+1}(t)=(1+h)u_n(t)-hu_0(t)-h\int_{t_0}^{t}\exp\left(\int_s^t B(\xi)d\xi\right)H(s)[A(s)+C(s)u_n^2(s)]ds \tag{10}$$

which is called the PIM expressed by sequence of second kind.

The $h\neq 0$ and $H(t)\neq 0$ denote the so-called auxiliary parameter and auxiliary function, respectively, which can be chosen in an efficient manner [10,12]. Accordingly, the successive approximations $u_n(t), n\geq 1$ of the PIM expressed by sequence will be readily obtained by choosing all the above-mentioned parameters. Consequently, the exact solution can be obtained by using

$$u(t)=\lim_{n\to\infty}u_n(t) \tag{11}$$

It is easy to observe that here we are capable of determining the multiplier without using the variational theory applied to the VIM [6-8]. This can be considered as an obvious advantage of the PIM expressed by sequence over the VIM. Also, the parametric iteration schemes of (7) and (10) reasonably include the VIM.

THE PIM EXPRESSED BY SERIES

In the next, the PIM expressed by series, which is a new feature of the PIM, is made clear for solving Eq. (1). This method will provide the solution of Eq. (1) as a series of approximations. To explain its idea behind, we first define the solution u(t) of Eq. (1) by a series as follows:

$$u_n(t)=\sum_{k=0}^{n}v_k(t),\qquad u(t)=\sum_{k=0}^{\infty}v_k(t) \tag{12}$$

According to the definition (12), the nonlinear term N[u] can be generally expressed in the form [12]:

$$N[u]=N\left[\sum_{k=0}^{\infty}v_k\right]=\sum_{k=0}^{\infty}N_k(v_0,\cdots,v_k) \tag{13}$$

where, by simple operation,

$$N_0(v_0)=N[u_0] \tag{14}$$

$$N_k(v_0,\cdots,v_k)=N\left[\sum_{i=0}^{k}v_i(t)\right]-N\left[\sum_{i=0}^{k-1}v_i(t)\right] \tag{15}$$

As a logical approximation of the identity (13), we have

$$N[u]=N\left[\sum_{k=0}^{\infty}v_k\right]=\sum_{k=0}^{\infty}N_k(v_0,\cdots,v_k)=\sum_{k=0}^{\infty}\overline{N}_k(v_0,\cdots,v_k) \tag{16}$$

where the $\overline{N}_k(v_0,\cdots,v_k)$'s are selected from the terms $N_k(v_0,\cdots,v_k)$ so that the sum of subscripts of the components of v(t) of each term of $\overline{N}_k(v_0,\cdots,v_k)$ is equal to k. Here we suppose that all series are convergent. Let us rewrite (3) in the following iteration formula:

$$\begin{aligned}u_{n+1}(t)=u_n(t)-h\int_{t_0}^{t}\Lambda_{(t,s)}(H(s)\{L[u_{n-1}(s)]+N[u_{n-1}(s)]-g(s)\})ds\\ -h\int_{t_0}^{t}\Lambda_{(t,s)}(H(s)\{L[u_n(s)-u_{n-1}(s)]+(N[u_n(s)]-N[u_{n-1}(s)])\})ds\end{aligned} \tag{17}$$

But it is well known from (3) that

$$u_n(t) = u_{n-1}(t) - h\int_{t_0}^{t} \Lambda_{(t,s)}(H(s)\{L[u_{n-1}(s)] + N[u_{n-1}(s)] - g(s)\})ds \tag{18}$$

In view of (13), by substituting (18) into (17) and then applying $u_n(t) = \sum_{k=0}^{n} v_k(t)$ on both sides of resulted formulation yield

$$v_{n+1}(t) = v_n(t) - h\int_{t_0}^{t} \Lambda_{(t,s)}(H(s)\{L[v_n(s)] + N_n[v_0(s),...,v_n(s)]\})ds \tag{19}$$

In a similar manner, in view of (16), we will have

$$v_{n+1}(t) = v_n(t) - h\int_{t_0}^{t} \Lambda_{(t,s)}(H(s)\{L[v_n(s)] + \bar{N}_n[v_0(s),...,v_n(s)]\})ds \tag{20}$$

According to (12), $u_0 = v_0$ and $u_1 = v_0 + v_1$, thus the components v_0 and v_1 of the recursive relations (19) and (20) are determined from (18) as:

$$v_0(t) = u_0(t)$$

$$v_1(t) = -h\int_{t_0}^{t} \Lambda_{(t,s)}(H(s)\{R_1[v_0(s)]\})ds \tag{21}$$

where

$$R_1[v_0(s)] = v_0'(s) - B(s)v_0(s) - C(s)v_0^2(s) - A(s) \tag{22}$$

In the light of (19)-(22), we can readily express the recursive relation of (19) and (20) by the function χ in the form

$$v_0(t) = u_0(t)$$

$$v_n(t) = \chi_n v_{n-1}(t) - h\int_{t_0}^{t} \Lambda_{(t,s)}(H(s)\{R_n[v_{n-1}(s)]\})ds \tag{23}$$

$$R_n[v_{n-1}(s)] = v_{n-1}'(s) - B(s)v_{n-1}(s) - C(s)N_{n-1}(v_0(s),...,v_{n-1}(s)) - A(s)(1-\chi_n)$$

which is called the PIM expressed by series of first kind and

$$v_0(t) = u_0(t)$$

$$v_n(t) = \chi_n v_{n-1}(t) - h\int_{t_0}^{t} \Lambda_{(t,s)}(H(s)\{\bar{R}_n[v_{n-1}(s)]\})ds \tag{24}$$

$$\bar{R}_n[v_{n-1}(s)] = v_{n-1}'(s) - B(s)v_{n-1}(s) - C(s)\bar{N}_{n-1}(v_0(s),...,v_{n-1}(s)) - A(s)(1-\chi_n)$$

which is called the PIM expressed by series of second kind, where

$$\chi_n = \begin{cases} 0, & \text{for} \quad n \le 1 \\ 1, & \text{for} \quad n > 1 \end{cases} \tag{25}$$

Therefore, the solution of Eq. (1) can be obtained as an infinite series, i.e., $u(t) = \sum_{k=0}^{\infty} v_n(t)$ given by (23) and (24) [12]. It is interesting to point out that these recursive procedures can be useful for revealing the relation between the PIM and the existing approximate analytical methods such as the ADM, HPM, HAM and the method proposed in [11].

A NUMERICAL IMPLEMENTATION

To give a clear overview of the content of this study, a quadratic Riccati differential equation will be studied. This equation will be tested by the above-mentioned algorithms, (7) and (24), which will ultimately show the efficacy of these methods [12]. All the results obtained here are calculated by using the symbolic calculus software Maple 11.

For the sake of comparative purposes, we consider a quadratic Riccati differential equation as follows [2-8,12]:

$$u' = 1 + 2u - u^2, \qquad u(0) = 0 \tag{26}$$

with the following exact solution

$$u(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \tag{27}$$

In order to solve Eq. (26) by using the PIM expressed by sequence of first kind (7), we choose $\Lambda_{(t,s)} = -1$ (i.e., $\alpha_1(t) = 1$ and $\alpha_0(t) = 0$) and H(t) = 1. We, therefore, have the following PIM formula [12]:

$$u_{n+1}(t) = u_n(t) + h\int_0^t (u_n'(s) - 1 - 2u_n(s) + u_n^2(s))ds \tag{28}$$

Starting from $u_0(t) = t$, we will have the following few approximations for (26):

$$u_1(t) = \frac{1}{3}ht^3 - ht^2 + t$$

$$u_2(t) = \frac{1}{63}h^3t^7 - \frac{1}{9}h^3t^6 + h\left(\frac{2}{15}h + \frac{1}{5}h^2\right)t^5 - \frac{2}{3}h^2t^4 + \left(\frac{1}{3}h + h\left(\frac{1}{3} + h\right)\right)t^3 + [-h + h(-1-h)]t^2 + t \tag{29}$$

$$\vdots$$

Here we have plotted the valid region h of the fifth-order PIM expressed by sequence of first kind solution in Fig. 1. Also, the approximate solutions of the 5th-order PIM expressed by sequence of first kind when h = -1 and h = -0.9 can be observed in Fig. 2.

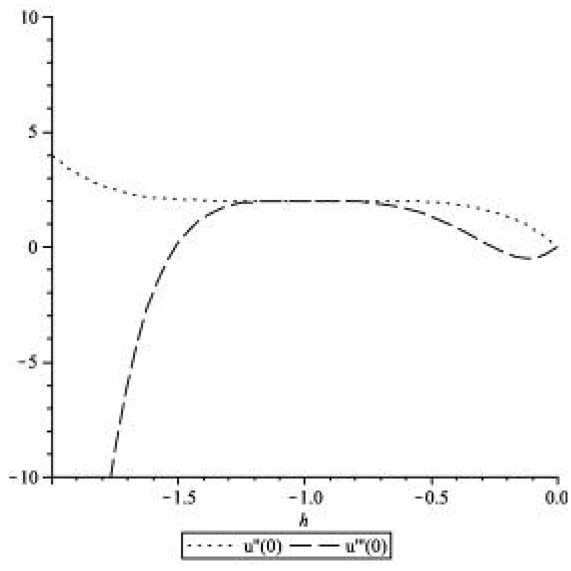

Fig. 1: The valid region of h by using the 5th-order PIM expressed by sequence of first kind solution for (26)

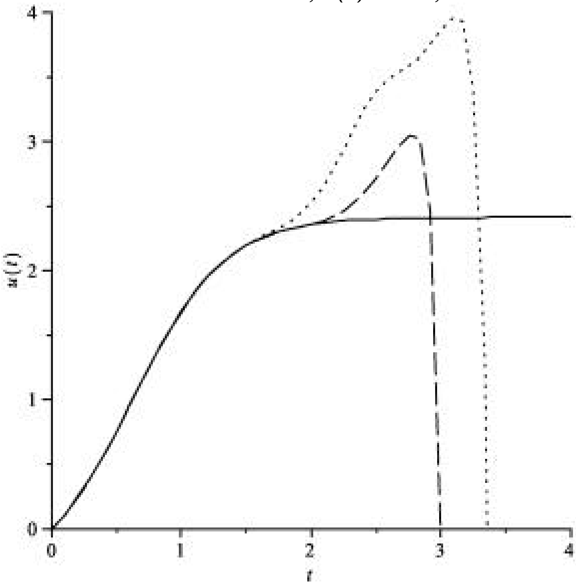

Fig. 2: Approximate solution of the 5th-order PIM expressed by sequence of first kind for (26) where Solid line: Exact, Dot line: h = -1 (i.e., the VIM) and Dash line: h = -0.9

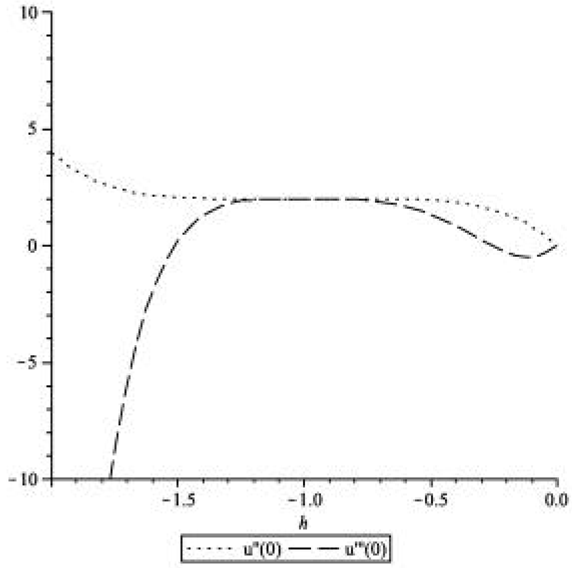

Fig. 3: The valid region of h by using the 5th-order PIM expressed by series of second kind solution for (26)

Now, according to (24), we will have the following few approximations of the PIM expressed by series of second kind for (26):

$$v_0(t) = t$$

$$v_1(t) = \frac{1}{3}ht^3 - ht^2 \tag{30}$$

$$v_2(t) = \frac{2}{15}h^2t^5 - \frac{2}{3}h^2t^4 + \left(\frac{1}{3}h + h^2\right)t^3 - h(1+h)t^2$$

$$\vdots$$

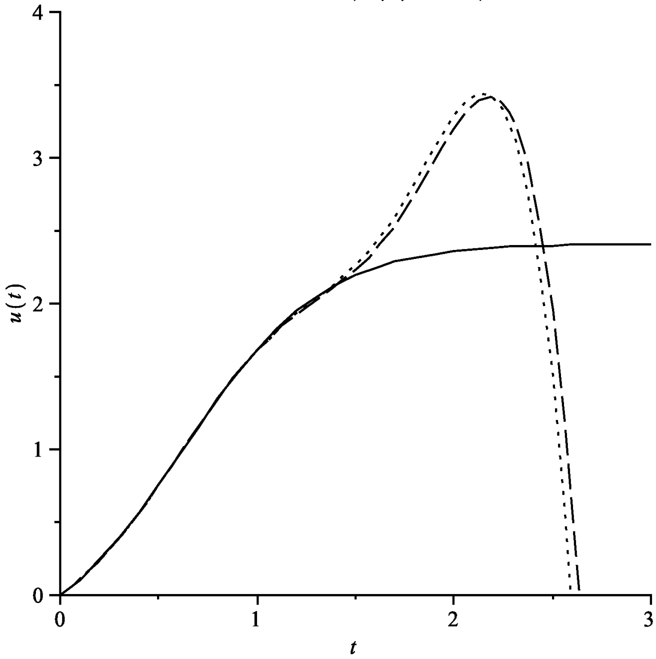

Fig. 4: Approximate solution of the 5th-order PIM expressed by series of second kind for (26) where Solid line: Exact, Dot line: h = -1 (i.e., the ADM and HPM) and Dash line: h = -0.9 (i.e., the HAM)

The valid region h and the approximate solution of the 5th-order PIM expressed by series of second kind can be seen in Fig. 3 and 4.

From the results proposed here, it is easy to conclude that the PIM proposed in this work could lead to a promising analytical method for solving nonlinear ordinary differential equations.

CONCLUDING COMMENTS

In this paper, the parametric iteration method (PIM) were applied to obtain approximate analytical solution for Riccati differential equations (RDEs). It was shown that the PIM is capable of providing the solution both as a sequence and as a series and logically includes some previous approximate analytical methods. This method provides us the convinent way to control the convergence rate of the solution. The numerical results reveal that the method presented in this paper is easy to implement. Moreover, it can further be employed easily to accurately solve other nonlinear ordinary differential equations.

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