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# New Exact Solutions of KdV Equation in an Elastic Tube Filled with a Variable Viscosity Fluid

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**Abstract:** In this paper, treating the arteries as a prestressed thin walled elastic tube with a stenosis and the blood as a Newtonian fluid with variable viscosity. Using the perturbation method, the KdV equation with variable coefficients are obtained.With the aid of the coordinate transformation, the reduced equation admits a progressive wave solution with variable wave speed. An extended tanh function method is used with a computerized symbolic computation for constructing the new exact travelling wave solutions of the reduced equation. The main idea of this method is to take full advantage of the Riccati equation which has more new solutions. The variations of radial displacement, fluid pressure with the distance parameter and speed wave are obtained. The method is straightforward and concise and its applications is promising.

**Key words:** KdV equation in an elastic tube filled with a variable viscosity fluid . Tanh function method . exact solutions

## INTRODUCTION

The applications in arterial mechanics, the propagation of pressure pulses in fluid filled distensible tubes has been studied by several researchers [1, 2]. Most of the works on wave propagation in compliant tubes have considered small amplitude waves ignoring the nonlinear effects and focused on the dispersive character of waves [3, 4]. The propagation of finite amplitude waves in fluid filled elastic or viscoelastic tubes has been examined, for instance, by Rudinger [5], Anliker *et al*. [6] and Tait and Moodie [7] by using the method of characteristics, in studying the shock formation. On the other hand, the propagation of small-but-finite amplitude waves in distensible tubes has been investigated by Johnson [8], Hashizume [9], Yomosa [10] and Demiray *et al*. [11-13].

The investigations of the travelling wave solution of nonlinear equations play an important role in the study of nonlinear physical phenomena. Amounts of mathematical models can be described by nonlinear equations, especially some basic equations in physics and mechanics. As a results, the reserach on exact solutions of nonlinear evolution equations becomes more and more important, such as [14-36].

In this paper, we shall study governing equations of an elastic tube filled with a Newtonian fluid of variable viscosity. Such a combination of a solid and a fluid is considered to be a model for blood flow in arteries, the forced KdV equation with variable coefficients as the evolution equation. The reduced equation is solved analytically by means of the extended tanh-function method. In Section 2, we present the govering equation in elastic tube filled with a variable viscosity fluid. The perturbation technique is employed to derive forced KdV equation with variable coefficients. In Section 3, we simply provide the mathematical framework of the improved tanh-function method. In Section 4, the proposed method is applied to solve the reduced equation. Finally, conclusion and discussion are given.

## FORMULATING THE KDV EQUATION IN AN ELASTIC TUBE FILLED WITH A VARIABLE VISCOSITY FLUID

The KdV equation in an elastic tube filled with a variable coefficients reads [13]

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$$\frac{\partial U}{\partial T}+\mu_1 U\frac{\partial U}{\partial \xi}-\mu_2\frac{\partial^2 U}{\partial \xi^2}+\mu_3\frac{\partial^3 U}{\partial \xi^3}+\mu_4(\tau)\frac{\partial U}{\partial \xi}=\mu(\tau) \tag{1}$$

Eq. (4.1), represents an elastic tube filled with a Newtonian fluid of variable viscosity. Such a combination of a solid and a fluid is considered to be a model for blood flow in arteries, where the coefficients $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$, $\mu_4(\tau)$ and $\mu(\tau)$ are defined by [13]

$$\begin{gathered}\mu_1=\frac{5}{2\lambda_\theta}+\frac{\beta_2}{\beta_1},\mu_2=\frac{\upsilon}{2c},\mu_3=\frac{m}{4\lambda_z}+\frac{\lambda_\theta^2}{16}-\frac{\beta_2}{2\beta_1}\\ \mu_4(\tau)=\frac{\lambda_\theta\gamma_2}{\beta_1}G(\tau)-[\frac{\beta_2}{\beta_1}+\frac{1}{2\lambda_\theta}]g(\tau),\mu(\tau)=\frac{g'(\tau)}{2}-\frac{\lambda_\theta\gamma_1}{2\beta_1}G'(\tau)\end{gathered} \tag{2}$$

By using the progressive wave solution to solve Eq.(1).For simplicity, we introduce the following new independent variable $V$ as:

$$U(\xi,\tau)=V(\xi,\tau)+\int_0^\tau \mu(s)ds=V(\xi,\tau)+\frac{1}{2}[g(\tau)-\frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)] \tag{3}$$

Inserting Eq.(3) into (1),yields

$$\frac{\partial V}{\partial \tau}+\mu_1 V\frac{\partial V}{\partial \xi}-\mu_2\frac{\partial^2 V}{\partial \xi^2}+\mu_3\frac{\partial^3 V}{\partial \xi^3}+\frac{\partial V}{\partial \xi}[\frac{\mu_1}{2}\left(g(\tau)-\frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right)+\mu_4(\tau)] \tag{4}$$

Making use the following coordinate transformation as

$$\tau=\tau,\xi'=\xi-\int_0^\tau[\frac{\mu_1}{2}(g(s)-\frac{\lambda_\theta\gamma_1}{\beta_1}G(s))+\mu_4(s)]ds \tag{5}$$

For new coordinate system, the evolution equation reduces to the conventional KdV equation as follows

$$\frac{\partial V}{\partial \tau'}+\mu_1 V\frac{\partial V}{\partial \xi'}-\mu_2\frac{\partial^2 V}{\partial \xi'^2}+\mu_3\frac{\partial^3 V}{\partial \xi'^3}=0 \tag{6}$$

Introducing the phase function $\eta$ is defined by

$$\eta=\alpha(\xi'-k\tau') \tag{7}$$

Where k and $\alpha$ are constants to determined later. Inserting Eq.(7) into (6),we have

$$-kV(\eta)+\frac{1}{2}\mu_1 V^2(\eta)-\mu_2\alpha V'(\eta)+\mu_3\alpha^2 V''(\eta)=0 \tag{8}$$

With the knowledge of the variation of radial displacement $U(\eta)$, allow us to calculate directly some physical parameter of special interest in physics,namely,the fluid pressure P with the distance parameter, wave speed $\upsilon_p$ and the variation of the viscosity $\gamma(x)$ as follows [13]

$$P=\beta_1[U(\eta)-g(\tau)]+\gamma_1\lambda_\theta G(\tau) \tag{9}$$

$$\gamma(x)=1-\frac{x}{\lambda_\theta-\varepsilon g(t)+U(\eta)},\upsilon_p=\frac{d\tau}{d\xi} \tag{10}$$

where

$$\beta_1 = \frac{2c^2}{\lambda_\theta}, \varepsilon$$

is a small parameter measuring the weakness of nonlinearity and dispersion and c is the scale parameter.

## METHODOLOGY

We give a brief description of the extended tanh function method. For a given the system of nonlinear evolution equations, say, in two variables x and t

$$\phi(u, \upsilon, u_t, \upsilon_t, u_x, \upsilon_x, u_{tt}, \upsilon_{tt}, u_{xx}, \upsilon_{xx}........) = 0 \tag{11}$$

$$\psi(u, \upsilon, u_t, \upsilon_t, u_x, \upsilon_x, u_{tt}, \upsilon_{tt}, u_{xx}, \upsilon_{xx}........) = 0 \tag{12}$$

We can seek their travelling wave solutions as follows

$$u(x,t) = u(\xi),\ \upsilon(x,t) = \upsilon(\xi), \quad \xi = x \pm ct$$

which are of important physical significance, c is constant to be determined later. Then system (4.11) and (4.12) reduces to

$$\phi_0(u, \upsilon, u_\xi, \upsilon_\xi, u_{\xi\xi}, \upsilon_{\xi\xi}, u_\xi, \upsilon_{\xi\xi}.....) = 0 \tag{13}$$

$$\psi_0(u, \upsilon, u_\xi, \upsilon_\xi, u_{\xi\xi}, \upsilon_{\xi\xi}, u_\xi, \upsilon_{\xi\xi}.....) = 0 \tag{14}$$

Introducing a new independent variables in the form

$$Y = \tanh(\mu\xi) \ \text{ or } \ Y = \coth(\mu\xi), \quad \xi = x \pm ct \tag{15}$$

that leads to the change of derivatives

$$\begin{aligned}
\frac{d}{d\xi} &= \mu\left(1 - Y^2\right)\frac{d}{dY} \\
\frac{d^2}{d\xi^2} &= -2\mu^2\left(1 - Y^2\right)\frac{d}{dY} + \mu^2\left(1 - Y^2\right)^2\frac{d^2}{dY^2} \\
\frac{d^3}{d\xi^3} &= 2\mu^3\left(1 - Y^2\right)\left(3Y^2 - 1\right)\frac{d}{dY} - 6\mu^3 Y\left(1 - Y^2\right)^2\frac{d^2}{dY^2} + \mu^3\left(1 - Y^2\right)^3\frac{d^3}{dY^3}
\end{aligned} \tag{16}$$

In the context of tanh function method, many authors [25-27] used the ansatz

$$u(\xi) = \sum_{i=0}^{M} a_i Y^i(\xi), \upsilon(\xi) = \sum_{i=0}^{N} c_i Y^i(\xi) \tag{17}$$

In order to construct more general,it is reasonable to introduce the following ansatz [28]

$$u(\xi) = \sum_{i=0}^{M} a_i Y^i(\xi) + \sum_{i=0}^{M} b_i Y^{-i}(\xi)$$

$$u(\xi)=\sum_{i=0}^{N}c_iY^i(\xi)+\sum_{i=0}^{M}d_iY^{-i}(\xi) \tag{18}$$

in which $a_i, b_i (i=0,1,...M)$ and $c_i, d_i (i=0,1,..N)$ are all real constants to be determined later, the balancing numbers M and N are positive integers which can be determined by balancing the highest order derivative terms with highest power nonlinear terms in Eqs.(13) and (14).We substitute anzatz (18) or (17) into Eqs.(13) and (14) with computerized symbolic computation, equating to zero the coefficients of all power $Y^{\pm i}(\xi)$ yields a set of algebraic equations for $a_i$, $b_i$, $c_i$, $d_i$ $\alpha$ and $\mu$.

## NEW EXACT SOLUTIONS OF THE KDV EQUATION WITH VARIABLE COEFFICIENTS

The main goal here is to solve the reduced Eq. (8) using the extended tanh method, we suppose that the solution can be expressed by

$$V(\eta)=\sum_{i=0}^{M}c_iY^i(\eta)+\sum_{i=1}^{M}b_iY^{-i}(\eta)$$

Balancing the highest linear terms with the highest nonlinear terms in
Eq. (8),we can found M = 2. Therefore, we assume the solution of Eq.(8) can be expressed as

$$V(\eta)=c_0+c_1Y(\eta)+c_2Y^2(\eta)+b_1Y^{-1}(\eta)+b_2Y^{-2}(\eta) \tag{19}$$

where $c_0$, $c_1$, $c_2$, $b_1$ are to be determined later. Substituting Eqs.(19) into Eq.(8) along with Eq.(16),and setting each coefficients of $Y^i(\eta)$ to zero, we can deduce the following set of algebraic polynomials for $c_0$, $c_1$, $c_2$, $b_1$, $b_2$, k and a. Solving the system of algebraic equations with the aid of Maple and Wu-Elemination method, we can distinguish three different cases, namely, as follows

**Case (1)**

$$c_0=\frac{3\mu_2^2}{10\mu_3\mu_1},c_2=-\frac{3\mu_2^2}{100\mu_3\mu_1},c_1=\frac{3\mu_2^2}{25\mu_3\mu_1},b_1=\frac{3\mu_2^2}{25\mu_3\mu_1},b_2=-\frac{3\mu_2^2}{100\mu_3\mu_1} \tag{20}$$

**Case (2)**

$$c_0=-\frac{3\mu_2^2}{25\mu_3\mu_1},c_2=-\frac{3\mu_2^2}{25\mu_3\mu_1},\mu=\mu,c_1=\frac{6\mu_2^2}{25\mu_3\mu_1},b_1=0,b_2=0 \tag{21}$$

**Case (3)**

$$c_0=\frac{9\mu_2^2}{25\mu_3\mu_1},c_2=0,c_1=0,b_1=-\frac{6\mu_2^2}{25\mu_3\mu_1},\mu=\mu,b_2=-\frac{3\mu_2^2}{25\mu_3\mu_1} \tag{22}$$

In view of case(1) and Eq.(3), we have a new exact travelling wave solution of Eq.(1) with variable coefficients, fluid pressure P with the distance parameter, wave speed $\upsilon_p$ and the variation of the viscosity $\gamma(x)$ as follows

$$U_1(\eta)=\frac{3\mu_2^2}{10\mu_3\mu_1}+\frac{3\mu_2^2\tanh(\mu\eta)}{25\mu_1\mu_3}+\frac{3\mu_2^2}{25\mu_1\mu_3\tanh(\mu\eta)}-\frac{3\mu_2^2\tanh^2(\mu\eta)}{100\mu_1\mu_3}-\frac{3\mu_2^2}{100\mu_1\mu_3\tanh^2(\mu\eta)}+\frac{1}{2}\left[g(\tau)-\frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right] \tag{23}$$

$$U_2(\eta)=\frac{3\mu_2^2}{10\mu_3\mu_1}+\frac{3\mu_2^2\coth(\mu\eta)}{25\mu_1\mu_3}+\frac{3\mu_2^2}{25\mu_1\mu_3\coth(\mu\eta)}-\frac{3\mu_2^2\coth^2(\mu\eta)}{100\mu_1\mu_3}-\frac{3\mu_2^2}{100\mu_1\mu_3\coth^2(\mu\eta)}+\frac{1}{2}\left[g(\tau)-\frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right] \tag{24}$$

$$P=\beta_1\left[U_i(\eta)-g(\tau)\right]+\gamma_1\lambda_\theta G(\tau),\quad i=1,2 \tag{25}$$

$$\gamma(x) = 1 - \frac{x}{\lambda_\theta - \varepsilon g(\tau) + U_i(\eta)}, \quad i = 1,2 \tag{26}$$

$$\eta = \alpha\left[\xi - a\tau - (\frac{3}{4\lambda_\theta} - \frac{\beta_2}{2\beta_1})g(\tau) - \frac{\lambda_\theta}{\beta_1}(\gamma_2 - \frac{\mu_1\gamma_1}{2}G(\tau)\right] \tag{27}$$

$$\upsilon_p = \frac{d\xi}{d\eta} = \frac{1}{a + \left(\frac{3}{4\lambda_\theta} - \frac{\beta_2}{2\beta_1}\right)g(\tau) + \frac{\lambda_\theta}{\beta_1}(\gamma_2 - \frac{\mu_1\gamma_1}{2})G(\tau)} \tag{28}$$

$$\alpha = -\frac{\mu_2}{20\mu_3\mu}, \qquad a = \frac{6\mu_2^2}{25\mu_3} \tag{29}$$

According to case(2) and Eq.(3), admits to new exact travelling wave solution of Eq.(1),fluid pressure P with the distance parameter, wave speed $\upsilon_p$ and the variation of the viscosity $\gamma(x)$ as follows

$$U_3(\eta) = -\frac{3\mu_2^2}{25\mu_3\mu_1} + \frac{6\mu_2^2\coth(\mu\eta)}{25\mu_1\mu_3} - \frac{3\mu_2^2\coth^2(\mu\eta)}{100\mu_1\mu_3} + \frac{1}{2}\left[g(\tau) - \frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right] \tag{30}$$

$$U_4(\eta) = -\frac{3\mu_2^2}{25\mu_3\mu_1} + \frac{6\mu_2^2\tanh(\mu\eta)}{25\mu_1\mu_3} - \frac{3\mu_2^2\tanh^2(\mu\eta)}{100\mu_1\mu_3} + \frac{1}{2}\left[g(\tau) - \frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right] \tag{31}$$

$$P = \beta_1[U_i(\eta) - g(\tau)] + \gamma_1\lambda_\theta G(\tau), \qquad i = 3,4 \tag{32}$$

$$\gamma(x) = 1 - \frac{x}{\lambda_\theta - \varepsilon g(\tau) + U_i(\eta)}, \quad i = 3,4 \tag{33}$$

$$\eta = \alpha\left[\xi - a\tau - (\frac{3}{4\lambda_\theta} - \frac{\beta_2}{2\beta_1})g(\tau) - \frac{\lambda_\theta}{\beta_1}(\gamma_2 - \frac{\mu_1\gamma_1}{2})G(\tau)\right] \tag{34}$$

$$\upsilon_p = \frac{1}{a + (\frac{3}{4\lambda_\theta} - \frac{\beta_2}{2\beta_1})g(\tau) + \frac{\lambda_\theta}{\beta_1}(\gamma_2 - \frac{\mu_1\gamma_1}{2})G(\tau)} \tag{35}$$

$$\alpha = -\frac{\mu_2}{10\mu_3\mu}, \qquad a = -\frac{6\mu_2^2}{25\mu_3} \tag{36}$$

By means of Eqs.(22) and (3),we have new travelling wave solution of Eq.(1),with fluid pressure, wave speed and the variation of the viscosity as follows

$$U_5(\eta) = \frac{9\mu_2^2}{25\mu_3\mu_1} - \frac{6\mu_2^2}{25\mu_1\mu_3\tanh(\mu\eta)} - \frac{3\mu_2^2}{25\mu_1\mu_3\tanh^2(\mu\eta)} + \frac{1}{2}\left[g(\tau) - \frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right] \tag{37}$$

$$U_6(\eta) = \frac{9\mu_2^2}{25\mu_3\mu_1} - \frac{6\mu_2^2}{25\mu_1\mu_3\tanh(\mu\eta)} - \frac{3\mu_2^2}{25\mu_1\mu_3\tanh^2(\mu\eta)} + \frac{1}{2}\left[g(\tau) - \frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right] \tag{38}$$

$$P = \beta_1[U_i(\eta) - g(\tau)] + \gamma_1\lambda_\theta G(\tau), \qquad i = 5,6 \tag{39}$$

$$\gamma(x) = 1 - \frac{x}{\lambda_\theta - \varepsilon g(\tau) + U_i(\eta)}, \quad i = 5,6 \tag{40}$$

$$\eta = \alpha\left[\xi - a\tau - (\frac{3}{4\lambda_\theta} - \frac{\beta_2}{2\beta_1})g(\tau) - \frac{\lambda_\theta}{\beta_1}(\gamma_2 - \frac{\mu_1\gamma_1}{2})G(\tau)\right] \tag{41}$$

$$\upsilon_p = \frac{1}{a + (\frac{3}{4\lambda_\theta} - \frac{\beta_2}{2\beta_1})g(\tau) + \frac{\lambda_\theta}{\beta_1}(\gamma_2 - \frac{\mu_1\gamma_1}{2})G(\tau)} \tag{42}$$

$$\alpha = \frac{\mu_2}{10\mu_3\mu}, \qquad a = \frac{6\mu_2^2}{25\mu_3} \tag{43}$$

## CONCLUSION

In summary, treating the arteries as a prestressed thin walled elastic tube with a stenosis and the blood as a Newtonian fluid with variable viscosity is studied. With the aid of the perturbation method, the KdV equation with variable coefficients as the evolution equation are obtained. By using the coordinate transformation, this type of evolution equation admits a progressive wave solution with variable wave speed. An extended tanh function method by using ansatz Eqs.(16) and (18) with a computerized symbolic computation for constructing the exact travelling wave solutions of the forced KdV equation with variable coefficients.

The proposed method is more effective and simple than other methods and a lot of solutions can be obtained in the same time. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation on a computer.

Finally, it is worthwhile to mention that the proposed method is reliable and effective and gives more solutions. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear partial differential equations arising in mathematical physics.

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