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# New Applications of the Modified Variational Iteration Method

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**Abstract:** In this work, we introduce new applications of Modified Variational Iteration Method (MVIM). The MVIM is used for solving Linear and nonlinear Schrödinger equations, the modified Kawahara equation and the Klien-Gordon-Type equation. A comparison between Variational Iteration Method (VIM) and MVIM shows that the MVIM numerical results convergences more rapidly than VIM numerical results. The MVIM method is capable of greatly reducing the size of calculations at the same time it maintains high accuracy of the numerical solution. Furthermore, the MVIM does not require a large capacity of computer memory and it doesn't take more time like VIM. The method is very simple and easy.

**Key words:** Nonlinear differential equation . Schrödinger equation . modified variational iteration method . variational iteration method . modified Kawahara equation . Klien-Gordon-type equation

## INTRODUCTION

The investigations of the exact solutions of nonlinear evolution equations play an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The exact solution, if available, of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions [1-4]. A broad class of analytical solutions methods and numerical solutions methods were used in to handle these problems such as piecewise analytic method [5-7], Backlund transformation [8], Hirota's bilinear method [9, 10], symmetry method [11], the inverse scattering transformation [12], the tanh method [13, 14], the Adomian decomposition method [15-17] and other asymptotic methods for strongly nonlinear equations [18].

The Variational Iteration Method (VIM) was proposed by "He" in 1997 [19-21]. It had been proved by many authors [21-33] to be a powerful mathematical tool for solving various types of nonlinear problems, which represent a plenty of modern science branches.

Abassy *et al*. tried to solve nonlinear partial differential equations using variational iteration method and found drawbacks in VIM. Abassy *et al*. introduced the modified variational iteration method which overcomes VIM drawbacks. The MVIM is used to give an approximate power series solutions for some well-known non-linear problems. It facilitates the computational work and minimizes it. This method can effectively improve the speed of convergence. Abassy *et al*. also proposed further treatments on MVIM results by using Padé approximants and Laplace transform. The treatment improves the convergence and gives the closed form solution in some cases, for more details see [34-38].

In this work we aim to introduce a new application of MVIM. MVIM is used to solve the Non-Linear Schrödinger (NLS) equation, modified Kawahara equation and the Klien-Gordon-Type equation.

The Non-Linear Schrödinger (NLS) equation has been established as a widely applicable model in various areas of physics as nonlinear optics, the theory of deep water waves, plasma physics, superconductivity, quantum mechanics, etc. A large number of researches conducted on the Nonlinear Schrödinger (NLS) equation. For example, its soliton solutions, conserved quantities, Bäcklund transformation, Darboux transformation and others have been discussed in Refs [12, 25, 39-51].

The modified Kawahara equation has wide applications in physics such as plasma waves, capillary-gravity water waves, water waves with surface tension, shallow water waves and so on [52-54].

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The nonlinear Klein-Gordon equation appears in many types of nonlinearities. It plays a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory [55, 56].

This paper demonstrates that MVIM solves the drawbacks in VIM through applying the method to nonlinear equations. The method is quite straightforward to write computer codes using Mathematica. It is demonstrated that the MVIM numerical results converge more rapidly than VIM numerical results.

## MODIFIED VARIATIONAL ITERATION METHOD

Consider the general non-linear initial value problem

$$\begin{gathered} Lu(x,t)+Ru(x,t)+Nu(x,t)=0,\ u(x,0)=f_0(x),\ \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=f_1(x) \\ \vdots \\ \left.\frac{\partial^{s-1}u(x,t)}{\partial t^{s-1}}\right|_{t=0}=f_{s-1}(x) \end{gathered} \tag{1}$$

where $L=\frac{\partial^s}{\partial t^s},\ s=1,2,3,\ldots$ is the highest partial derivative with respect to t, R is a linear operator and Nu(x, t) is the nonlinear term. Ru(x, t) and Nu(x, t) are free of partial derivative with respect to t.

MVIM [34, 35] is used for solving (1) which the following iteration formula is used

$$U_{n+1}(x,t)=U_n(x,t)+\int_0^t \lambda(\tau)\{R(U_n-U_{n-1})+(G_n-G_{n-1})\}d\tau \tag{2}$$

where $\lambda(\tau)$ is called a general Lagrange multiplier and equals $\frac{-(t-\tau)^{(s-1)}}{(s-1)!}$, which is identified optimally via variational theory, $G_n(x,t)$ is a polynomial of degree (s(n+1)-1) and is obtained by

$$NU_n(x,t)=G_n(x,t)+O(t^{s(n+1)})$$

The iteration formula (2) can be solved iteratively using

$$U_{-1}=0,\ U_0=f_0(x)+f_1(x)t+\ldots+\frac{f_{s-1}(x)}{(s-1)!}t^{s-1}$$

to obtain an approximate power series solution for equation (1).

MVIM gives an approximate series solution that converges to equations (1) closed form solution in the neighborhood of initial points.

## APPLICATION 1: SCHRÖDINGER EQUATION

In the following subsections, two case studies, one is linear and the other is nonlinear, are solved to illustrate the efficiency of MVIM.

The linear one takes the form

$$u_t+i\,u_{xx}=0,\ \ u(x,0)=f(x),\ \ i^2=-1 \tag{3}$$

and the nonlinear Schrödinger equation takes the form

$$u_t-i\,u_{xx}-i\,q|u|^2\,u=0,\ u(x,0)=f(x) \tag{4}$$

**Case-study 1:** Consider the linear Schrödinger equation

$$u_t + iu_{xx} = 0, u(x,0) = 1 + \cosh(2x) \tag{5}$$

Solving (5) by MVIM where

$$Ru(x,t) = i\frac{\partial^2 u(x,t)}{\partial x^2}$$

Nu(x, t) = 0, s = 1 which leads to $\lambda = -1$ and the following iteration formula is used:

$$U_{n+1} = U_n - \int_0^t \{R(U_n - U_{n-1})\} d\tau \tag{6}$$

where $U_{-1} = 0$ and $U_0 = 1+\cosh(2x)$
The following results are obtained

$$\begin{aligned}
U_1 &= 1 + \cosh(2x) - 4it\cosh(2x) \\
U_2 &= 1 + \cosh(2x) - 4it\cosh(2x) - 8t^2\cosh(2x) \\
U_3 &= 1 + \cosh(2x) - 4it\cosh(2x) - 8t^2\cosh(2x) + \frac{32}{3}it^3\cosh 2x \\
U_4 &= 1 + \cosh(2x) - 4it\cosh(2x) - 8t^2\cosh(2x) + \frac{32}{3}it^3\cosh(2x) + \frac{32}{3}t^4\cosh(2x) \\
U_5 &= 1 + \cosh(2x) - 4it\cosh(2x) - 8t^2\cosh(2x) + \frac{32}{3}it^3\cosh(2x) + \frac{32}{3}t^4\cosh(2x) - \frac{128}{15}t^5\cosh(2x) \\
&\vdots
\end{aligned} \tag{7}$$

This is an approximate power series expansion which converges to equation (5) closed form solution

$$u(x,t) = 1 + \cosh(2x)e^{-4it} \tag{8}$$

Applying the VIM gives the same result.

**Case-study 2:** Consider the nonlinear Schrödinger equation

$$u_t - iu_{xx} - iq|u|^2 u = 0, u(x,0) = e^{ikx} \tag{9}$$

where $i = \sqrt{-1}$, q and k are constants.

Substituting by $u = u(x,t) = a(x,t) + ib(x,t)$ in equation (9) before solving it with MVIM, where a and b are real-valued continuous functions of x and t. It leads to the following coupled system of equations:

$$a_t + b_{xx} + q(a^2 + b^2)b = 0 \tag{10}$$

$$b_t - a_{xx} - q(a^2 + b^2)a = 0 \tag{10b}$$

and subjected to the following initial conditions

$$a(x,0) = \cos(kx), \quad b(x,0) = \sin(kx)$$

Equation (10) and **Error! Reference source not found.** is used to obtain solution for a and b respectively.
Applying MVIM to equation (10) gives us:

$$R1 = \frac{\partial^2 b(x,t)}{\partial x^2}, N1 = q(a^2 + b^3)a, \frac{\partial a}{\partial t}$$

leads to s = 1, $\lambda$ = -1 and using the following iteration formula

$$A_{n+1} = A_n - \int_0^t \{(B_n - B_{n-1})_{xx} + (GA_n - GA_{n-1})\} d\tau \tag{11}$$

Applying MVIM to equation **Error! Reference source not found.** gives us:

$$R2 = \frac{\partial^2 a(x,t)}{\partial x^2}, N2 = -q(a^2 + b^2)a, \frac{\partial b}{\partial t}$$

leads to s = 1, $\lambda$ = -1 and the following iteration formula is used:

$$B_{n+1} = B_n - \int_0^t \{-(A_n - A_{n-1})_{xx} + (GB_n - GB_{n-1})\} d\tau \tag{12}$$

Equation (11) and (12) are solved iteratively and simultaneously using

$$A_{-1} = 0, B_{-1} = 0, A_0 = \cos(kx), B_0 = \sin(kx)$$

$GA_n(x,t)$ and $GB_n(x,t)$ are calculated by

$$q\left(A_n(x,t)^2 + B_n(x,t)^2\right)B_n(x,t) = GA_n(x,t) + O(t^{n+1}) - q\left(A_n(x,t)^2 + B_n(x,t)^2\right)A_n(x,t) = GB_n(x,t) + O(t^{n+1})$$

The following results are obtained

$$\begin{aligned}
A_1 &= \cos(kx) + (k^2 - q)t\sin(kx) \\
B_1 &= \sin(kx) - (k^2 - q)t\cos(kx) \\
A_2 &= \cos(kx) + (k^2 - q)t\sin(kx) - \frac{(k^2 - q)}{2}t^2\cos(kx) \\
B_2 &= \sin(kx) - (k^2 - q)t\cos(kx) - \frac{(k^2 - q)^2}{2}t^2\sin(kx) \\
A_3 &= \cos(kx) + (k^2 - q)t\sin(kx) - \frac{(k^2 - q)^2}{2}t^2\cos(kx) - \frac{(k^2 - q)^3}{6}t^3\sin(kx) \\
B_3 &= \sin(kx) - (k^2 - q)t\cos(kx) - \frac{(k^2 - q)^2}{2}t^2\sin(kx) + \frac{(k^2 - q)^3}{6}t^3\cos(kx) \\
&\vdots
\end{aligned} \tag{13}$$

This solution is an approximate power series solution to the closed form solution of (9)

$$u(x,t) = e^{i(kx - (k^2 - q)t)} \tag{14}$$

Appling VIM leads to using the following iteration formulas

$$A_{n+1} = A_n - \int_0^t \left\{A_n(x,\tau)_\tau + B_n(x,\tau)_{xx} + q\left(A_n(x,\tau)^2 + B_n(x,\tau)^2\right)B_n(x,\tau)\right\} d\tau \tag{15}$$

$$B_{n+1} = B_n - \int_0^t \left\{B_n(x,\tau)_\tau - A_n(x,\tau)_{xx} - q\left(A_n(x,\tau)^2 + B_n(x,\tau)^2\right)A_n(x,\tau)\right\} d\tau \tag{16}$$

which are solved simultaneously and give the following results:

$$A_1 = \cos(kx) + (k^2 - q)t\sin(kx)$$
$$B_1 = \sin(kx) - (k^2 - q)t\cos(kx)$$
$$A_2 = \cos(kx) + (k^2 - q)t\sin(kx) - \frac{(k^2 - q)}{2}t^2\cos(kx) - \frac{1}{3}(k^2 - q)^2 q t^3 \sin(kx) - \frac{1}{4}(k^2 - q)^4 q\, t^4 \cos(kx)$$
$$B_2 = \sin(kx) - (k^2 - q)t\cos(kx) - \frac{(k^2 - q)^2}{2}t^2\sin(kx) + \frac{1}{3}(k^2 - q)^2 q\, t^3 \cos(kx) - \frac{1}{4}(k^2 - q)^3 q t^4 \sin(kx) \quad (17)$$
$$A_3 = \cos(kx) + (k^2 - q)t\sin(kx) - \frac{(k^2 - q)^2}{2}t^2\cos(kx) - \frac{(k^2 - q)^3}{6}t^3\sin(kx) - \frac{(k^2 - q)^2(q^2 - qk^2)}{12}t^4\cos(kx) + \ldots$$
$$B_3 = \sin(kx) - (k^2 - q)t\cos(kx) - \frac{(k^2 - q)^2}{2}t^2\sin(kx) + \frac{(k^2 - q)^3}{6}t^3\cos(kx) - \frac{(k^2 - q)^2(q^2 - qk^2)}{12}t^4\sin(kx) + \ldots$$
$$\vdots$$

Table 1 shows the consumed time in calculating $U_n(x,t)$ using MVIM and VIM. It is clear from Table 1, MVIM is very fast with respect to VIM. $U_6^{MVIM}(x,t)$ is calculated by MVIM while $U_2^{VIM}(x,t)$ is still being calculated by VIM. Figure 1 shows the absolute error between the closed form solution and $U_2(x,t)$ (MVIM and VIM). Figure 2 shows the absolute error between the closed form solution and $U_6(x,t)$ in MVIM and $U_3(x,t)$ in VIM. Figure 3 shows that the Absolute error is decreased as successive terms are calculated.

## APPLICATION 2: THE MODIFIED KAWAHARA EQUATION

Consider the modified Kawahara equation [52] given by

$$u_t + u_x + u^2u_x + pu_{xxx} + qu_{xxxxxx} = 0, u(x,0) = \frac{3p}{\sqrt{-10q}}\operatorname{sech}^2(kx) \quad (18)$$

Table 1: The time consumed in calculating $U_n(x,t)$ for NL Sequation (case-study 2) by VIM and MVIM using Mathematica 7 package

| | $U_1(x,t)$ | $U_2(x,t)$ | $U_3(x,t)$ | $U_4(x,t)$ | $U_5(x,t)$ | $U_6(x,t)$ |
|---|---|---|---|---|---|---|
| VIM | 0.141 | 2.438 | 52.375 | 290.781 | N/A | N/A |
| MVIM | 0.171 | 0.437 | 0.577 | 0.734 | 0.89 | 1.062 |

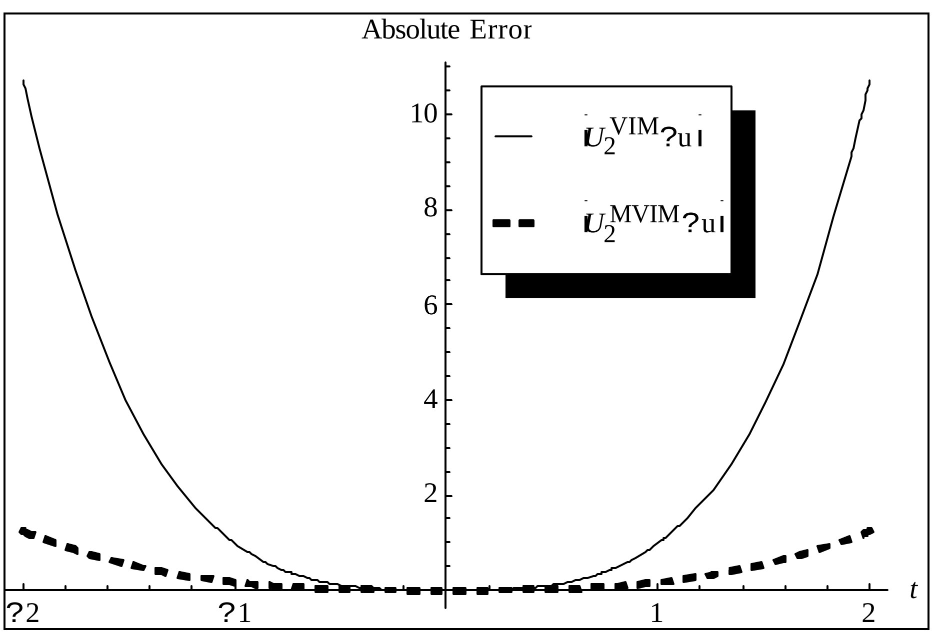


Fig. 1: The Absolute error between the exact solution of (case-study 2)and approximate solution $U_2(x,t)$ obtained by MVIM and VIM (x = 0, q = 2 and k = 1)

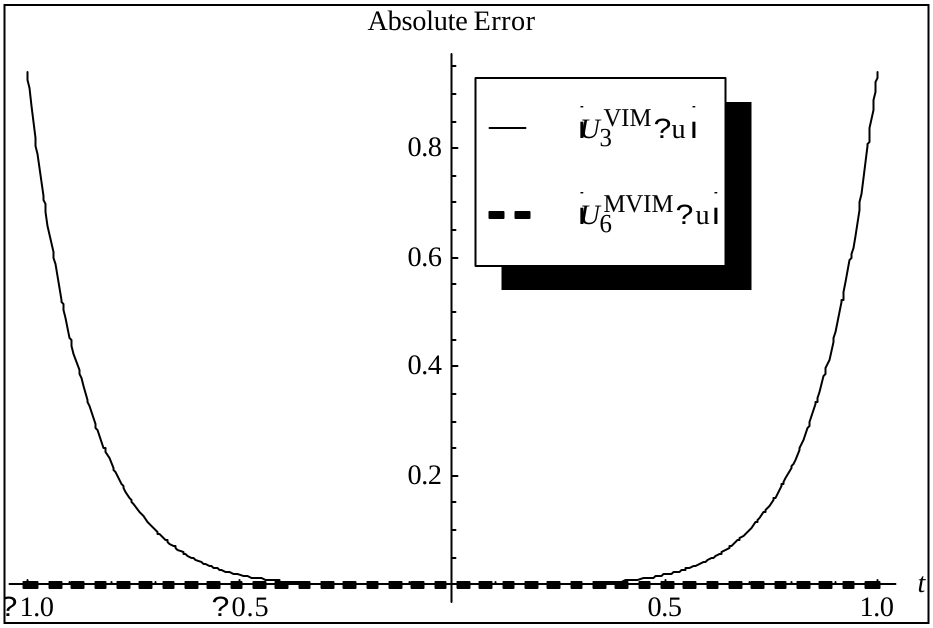


Fig. 2: The Absolute error between the exact solution of (case-study 2)and approximate solution $U_3^{VIM}(x,t)$ and $U_6^{MVIM}(x,t)$ (x = 0, q = 2 and k = 1)

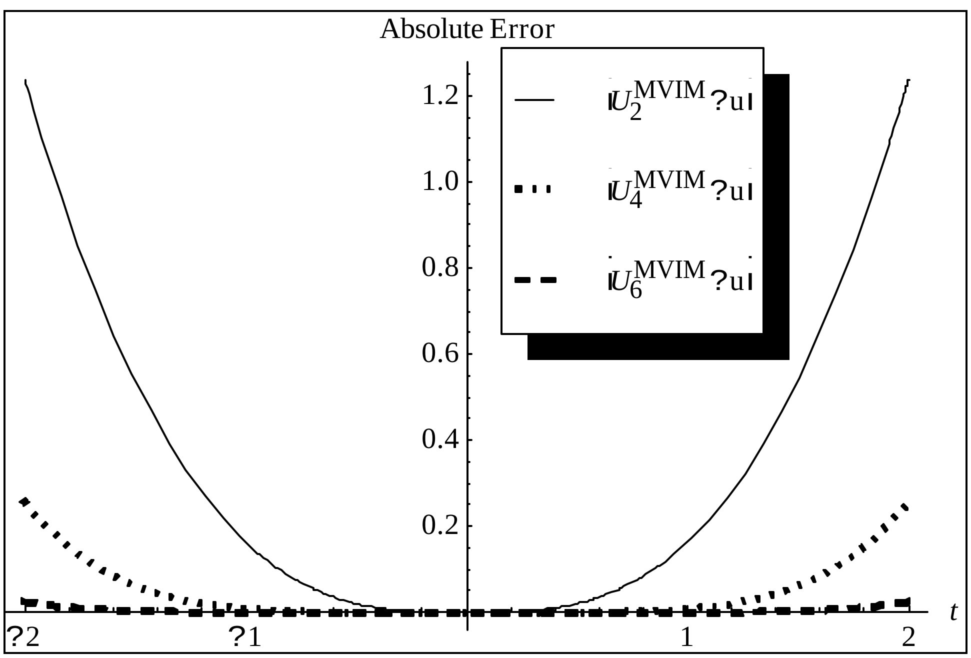


Fig. 3: The Absolute error between the exact solution of (case-study 2)and approximate solution $U_2^{MVIM}(x,t)$, $U_4^{MVIM}(x,t)$ and $U_6^{MVIM}(x,t)$ (x = 0, q = 2 and k = 1)

where p and q are nonzero real constants and $k=\frac{1}{2}\sqrt{\frac{-p}{5q}}$.

Solving equation (18) using MVIM [34, 35] we found that:

$Ru(x,t)= u_x + pu_{xxx} + qu_{xxxxxx}$, $Nu(x,t)=u^2u_x$, s = 1 which leads to $\lambda$ = -1 and the following iteration formula is used:

$$U_{n+1} = U_n - \int_0^t \left\{R\left(U_n - U_{n-1}\right) + (G_n - G_{n-1})\right\} d\tau \tag{19}$$

Where $U_{-1} = 0$, $U_0 = \frac{3p}{\sqrt{-10q}}\operatorname{sech}^2(kx)$, and $G_n(x,t)$ is a polynomial of degree n in t and is obtained from

$$\left(U_n(x,t)\right)^2 \left(U_n(x,t)\right)_x = G_n(x,t) + O(t^{n+1})$$

The following results are obtained

$$\begin{aligned}
U_1 &= U_0 + \frac{6p}{\sqrt{-10q}} a \tanh(kx)\operatorname{sech}^2(kx)\, t \\
U_2 &= U_1 + \frac{3p}{\sqrt{-10q}} a^2(-2+\cosh(2kx))\operatorname{sech}^4(kx)t^2 \\
U_3 &= U_2 + \frac{p}{\sqrt{-10q}} a^3 \,(-11\sinh(kx)+\sinh(3kx))\operatorname{sech}^5(kx)\, t^3 \\
U_4 &= U_3 + \frac{p}{\sqrt{-10q}} a^4(33-26\cosh(2kx)+\cosh(4kx))\operatorname{sech}^6(kx)t^4 \\
U_5 &= U_4 + \frac{p}{20\sqrt{-10q}} a^5 \,(302\sinh(kx)-57\sinh(3kx)+\operatorname{Sinh}(5kx))\operatorname{sech}^7(kx)\, t^5 \\
&\vdots
\end{aligned}$$

Where

$$a = \frac{25q-4p^2}{25q}k \;\text{ and }\; k = \frac{1}{2}\sqrt{\frac{-p}{5q}}\,.$$

This is an approximate power series expansion which converges to equation (18) closed form solution

$$u(x,t) = \frac{3p}{\sqrt{-10q}}\operatorname{sech}^2(k(x-at)) \tag{20}$$

Solving equation (18) using VIM we found that:

$$Ru(x,t) = u_x + pu_{xxx} + qu_{xxxxxx}, \quad Nu(x,t) = u^2u_x$$

and the following iteration formula is used:

$$U_{n+1} = U_n - \int_0^t \left\{\left(U_n(x,\tau)\right)_\tau + R\left(U_n(x,\tau)\right) + N\left(U_n(x,\tau)\right)\right\} d\tau \tag{21}$$

where $U_0 = \frac{3p}{\sqrt{-10q}}\operatorname{sech}^2(kx)$, The following results are obtained

$$\begin{aligned}
U_1^{VIM} &= U_0 + \frac{6p}{\sqrt{-10q}} a \tanh(kx)\operatorname{sech}^2(kx)t, \\
U_2^{VIM} &= U_0 + \frac{6p}{\sqrt{-10q}} a \tanh(kx)\operatorname{sech}^2(kx)t + \frac{3p}{\sqrt{-10q}} a^2(-2+\cosh(2kx))\operatorname{sech}^4(kx)\, t^2 + O(t^3) \\
U_3^{VIM} &= U_0 + \frac{6p}{\sqrt{-10q}} a \tanh(kx)\operatorname{sech}^2(kx)t + \frac{3p}{\sqrt{-10q}} a^2(-2+\cosh(2kx))\operatorname{sech}^4(kx)\, t^2 \\
&\quad + \frac{p}{\sqrt{-10q}} a^3 \left(-11\sinh(kx)+\sinh(3kx)\right)\operatorname{sech}^5(kx)t^3 + O(t^4) \\
&\vdots
\end{aligned} \tag{22}$$

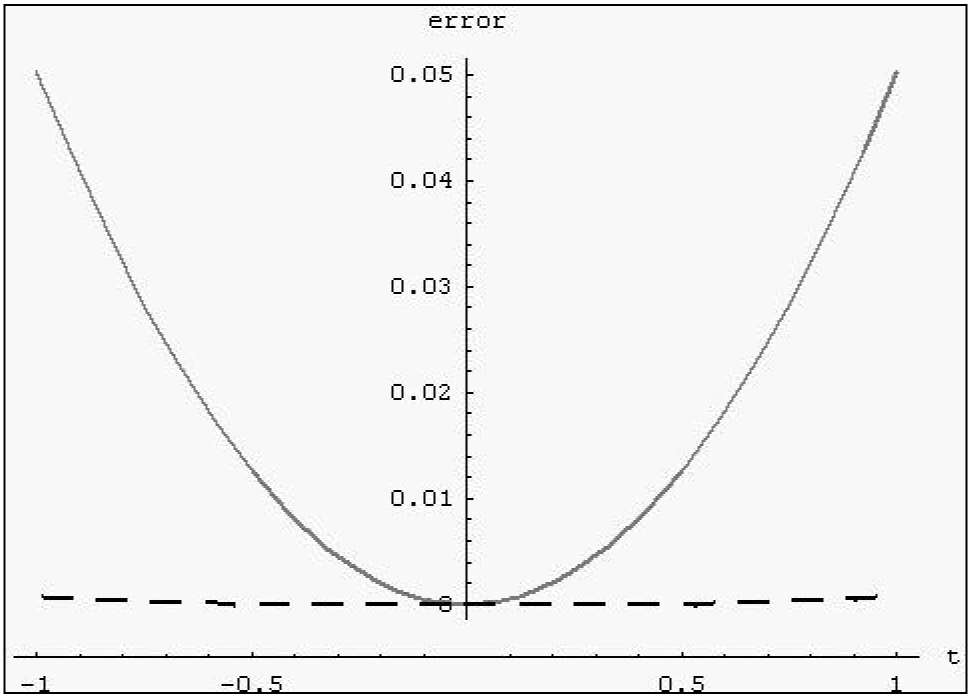


Fig. 4: Solid gray line represents the absolute error between $U_2^{VIM}$ and closed form solution (20) and dashed line represents the absolute error between $U_2^{MVIM}$ and closed form solution (21) (x = 0, p = 1, q = 1)

Where

$$a = \frac{25q - 4p^2}{25q}k \text{ and } k = \frac{1}{2}\sqrt{\frac{-p}{5q}}.$$

Comparing the results of VIM and MVIM, it is founded

$$\begin{aligned} U_1^{(VIM)} &= U_1^{(MVIM)} \\ U_2^{(VIM)} &= U_2^{(MVIM)} + O(t^3) \\ U_3^{(VIM)} &= U_3^{(MVIM)} + O(t^4) \\ &\vdots \end{aligned} \tag{23}$$

It is massive to write all the results of VIM in equation (23) and (24) so we write these terms in the form $O(t^n)$.

Analyzing equation (24), it is founded that all the terms in MVIM results contained in VIM results but there is other terms in VIM which deteriorate the convergence of VIM and take too much time in calculation. Figure 4 shows the absolute error.

**APPLICATION 3: THE NONLINEAR KLIEN-GORDON-TYPE EQUATION**

Consider the nonlinear Klien-Gordon-Type equation [55, 57]

$$
\begin{aligned}
&u_{tt}-a^2u_{xx}+\alpha u-\beta u^3=0\\
&u(x,0)=\sqrt{\frac{2\alpha}{\beta}}\operatorname{sech}(k(x))\\
&u_t(x,0)=\sqrt{2}ck\sqrt{\frac{2\alpha}{\beta}}\operatorname{sech}(k(x))\tanh(k(x))
\end{aligned}
\tag{24}
$$

where

$$k=\sqrt{\frac{\alpha}{a^2-c^2}},\alpha,\beta,a$$

and c are constants.
Solving equation (24) using MVIM [34, 38] we found that:
$Ru(x,t)=-a^2u_{xx}+\alpha u$, $Nu(x,t)=-\beta u^3$, s = 2 which leads to $\lambda$ = -(t-$\tau$) and the following iteration formula is used:

$$U_{n+1}=U_n-\int_0^t(t-\tau)\{R(U_n-U_{n-1})+(G_n-G_{n-1})\}d\tau \tag{25}$$

Where $U_{-1}=0$

$$U_0=\sqrt{\frac{2\alpha}{\beta}}\operatorname{sech}(k(x))+\sqrt{2}ck\sqrt{\frac{2\alpha}{\beta}}\operatorname{sech}(k(x))\tanh(k(x))t$$

and $G_n(x,t)$ is a polynomial of degree (2n+1) in t and is obtained from

$$-\beta(U_n(x,t))^3=G_n(x,t)+O(t^{2(n+1)})$$

The following results are obtained

$$
\begin{aligned}
&U_1=U_0+\frac{c^2}{12\sqrt{2}}k^2\sqrt{\frac{\alpha}{\beta}}(-15\cosh(kx)+3\cosh(3kx))\operatorname{sech}^4(kx)t^2+\frac{c^3}{12\sqrt{2}}k^3\sqrt{\frac{\alpha}{\beta}}(-23\sinh(kx)+3\sinh(3kx))\operatorname{sech}^4(kx)t^3\\
&U_2=U_1+\frac{c^4}{960\sqrt{2}}k^4\sqrt{\frac{\alpha}{\beta}}(770\cosh(kx)-375\cosh(3kx)+5\cosh(5kx))\\
&\qquad\operatorname{sech}^6(kx)t^4+\frac{c^5}{960\sqrt{2}}k^5\sqrt{\frac{\alpha}{\beta}}(1682\sinh(kx)-237\sinh(3kx)+\sinh(5kx))\operatorname{sech}^6(kx)t^5\\
&U_3=U_2+\frac{c^6}{161280\sqrt{2}}k^6\sqrt{\frac{\alpha}{\beta}}\operatorname{sech}^8(kx)t^6(-91035\cosh(kx)+68747\cosh(3kx)-5047\cosh(5kx)+7\cosh(7kx))+\\
&\qquad\frac{c^7}{161280\sqrt{2}}k^7\sqrt{\frac{\alpha}{\beta}}\operatorname{sech}^8(kx)t^7(-259723\sinh(kx)+60657\sinh(3kx)-2179\sinh(5kx)+\sinh(7kx))\\
&\qquad\vdots
\end{aligned}
\tag{26}
$$

Where $k=\sqrt{\frac{\alpha}{a^2-c^2}}$.
This is a power series expansion of the closed form solution

$$u(x,t)=\sqrt{\frac{2\alpha}{\beta}}\operatorname{sech}\left(\sqrt{\frac{\alpha}{a^2-c^2}}(x-ct)\right) \tag{27}$$

Solving equation (24) using VIM we found that:
$Ru(x,t)=-a^2u_{xx}+\alpha u$, $Nu(x,t)=-\beta u^3$ and the following iteration formula is used:

$$U_{n+1} = U_n - \int_0^t (t-\tau)\left\{\left(U_n(x,\tau)\right)_\tau + R\left(U_n(x,\tau)\right) + N\left(U_n(x,\tau)\right)\right\} d\tau \tag{28}$$

where

$$U_0 = \sqrt{\frac{2\alpha}{\beta}}\,\mathrm{sech}(k(x)) + \sqrt{2}ck\sqrt{\frac{2\alpha}{\beta}}\,\mathrm{sech}(k(x))\tanh(k(x))t$$

The following results are obtained

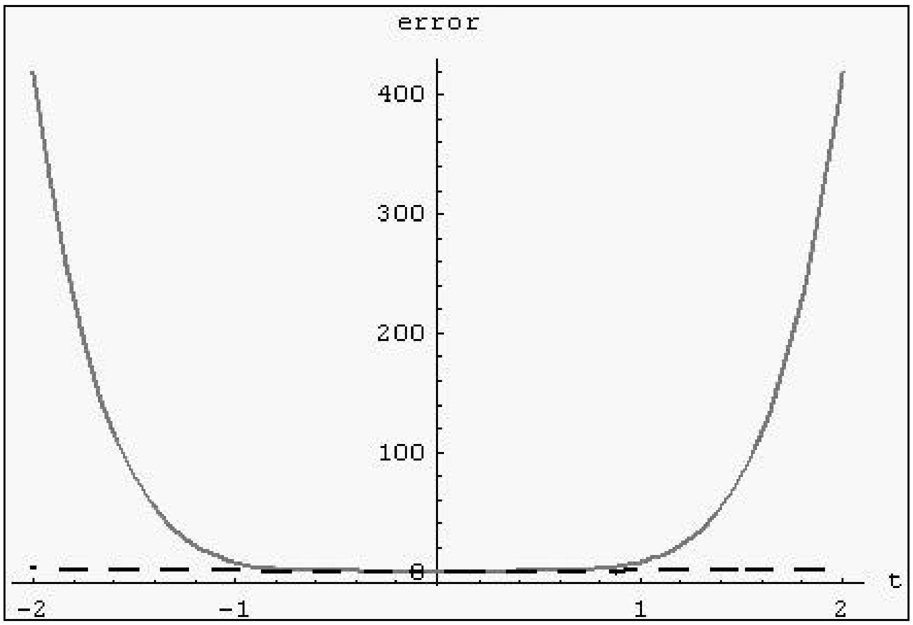


Fig. 5: Solid gray line represents the absolute error between $U_2^{VIM}$ and closed form solution (27) and dashed line represents the absolute error between $U_2^{MVIM}$ and closed form solution (27) $(\alpha = 1, \beta = 2, a = \sqrt{2}, c = 1, x = 0)$.

$$\begin{aligned} U_1^{(VIM)} &= U_1^{(MVIM)} + O(t^4) \\ U_2^{(VIM)} &= U_2^{(MVIM)} + O(t^6) \\ U_3^{(VIM)} &= U_3^{(MVIM)} + O(t^8) \\ &\vdots \end{aligned} \tag{29}$$

and Fig. 5 shows the absolute error.

## DISSECTION

Analyzing the results obtained by VIM, it has been observed that $U_n^{VIM}$ takes the following form:

$$U_n = B_n^0 + B_n^1 t + B_n^2 t^2 + \ldots + B_n^{s(n+1)-1} t^{s(n+1)-1} + B_n^{s(n+1)} t^{s(n+1)} + B_n^{s(n+1)+1} t^{s(n+1)+1} + \ldots \tag{30}$$

where $B_n^m$ is the coefficient of $t^m$. $B_n^m$ is settled and takes the same value for each $U_n^{VIM}$ when $m \le s(n+1)-1$ and is not settled and doesn't take the same value for each $U_n^{VIM}$ when $m > s(n+1)-1$.

We can depend on the settled terms. The non-settled terms we can not depend on them which deteriorates the approximate series solution of the variational iteration method and consume time and effort in calculation

By observing the results obtained by MVIM and VIM in the illustrative examples, we found that MVIM eliminates all the non-settled terms in VIM. It is faster in calculations and more convergent than VIM [34,35].

**SUMMARY**

In this work, the linear and the nonlinear Schrödinger equations, the nonlinear modified Kawahara equation and the nonlinear Klein-Gordon equation are efficiently handled by Modified variational iteration method. MVIM gives rapid convergent successive approximations and it is capable of greatly reducing the size of calculations. The MVIM is not faced with necessity of large computer memory and plenty time consumption like VIM. MVIM is an efficient method to handle nonlinear structure which gives exact series solution that converges to exact solution in the neighborhood of initial point.

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