Studies in Nonlinear Sciences 2 (4): 135-139, 2011
ISSN 2221-3910

Coincidence Theorems on Product FC-spaces

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Abstract: By applying the technique of continuous partition of unity, some new coincidence theorems for a better admissible mapping of a family of set-valued mappings defined on the product FC-spaces are proved under suitable conditions. The results presented in this paper unity and generalize some known coincidence theorems in recent literatures.

2000 MSC: 54H25 . 47H10 . 54C60

Key words: Better admissible mapping . coincidence theorems . FC-space

INTRODUCTION

In 1937, Von Neumann [1] established the famous coincidence theorem. Since then, the coincidence theorem was generalized in many directions. Browder [2] first proved some basic coincidence theorems for a pair of set-valued mappings in compact setting of topological vector spaces and gave some applications to minimax inequalities and variational inequalities. Recently, Ding [3] established some new coincidence theorems for a better admissible mapping on G-convex spaces by using the technique of a continuous partition of unity. In this paper, we will generalize these coincidence theorems on FC-spaces without convexity structure.

PRELIMINARIES

Let X and Y be two nonempty sets. We denote $\langle X\rangle$ and 2^Y the family of all nonempty finite subsets of X and the family of all subsets of Y, respectively. For each $A \in \langle X\rangle$, we denote $|A|$ the cardinality of A. Let Δ_n be the standard n-dimentional simplex with vertices $\{e_0, e_1, \ldots$�$e_n\}$. If J is a nonempty subset of $\{0,1,\ldots,n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j: j \in J\}$. A subset A of a topological space X is said to be compactly open (resp., compactly closed) in X if for each nonempty compact subset K of X, $A \cap K$ is open (resp., closed) in K. The following notions were introduced by Ding [4], For any given nonempty subset A of X, denote the compact interior and the compact closure of A, denoted by c int(A) and ���(A), as

$$\mathrm{cint}(A) = \bigcup\{B \subset X : B \subset A \text{ and } B \text{ is completely open in } X\}$$

$$\mathrm{ccl}(A) = \bigcap\{B \subset X : A \subset B \text{ and } B \text{ is completely closed in } X\}$$

It is easy to see that c int(A) is compactly open in X and ccl(A) is compactly closed in X. It is clear that a subset A of X is compactly open (resp., compactly closed) in X if and only if A = c int (A) (resp., A = c ccl (A)).
The following notion was introduced by Ding [5].

Definition 2.1: (Y, φ_N) is said to be a finitely continuous topological space (for short, FC-space) if Y is a topological space such that for each

$$N = \{y_0, y_1, \ldots, y_n\} \in \langle Y \rangle$$

there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset D of (Y, φ_N) is said to be an FC-subspace of Y if for each

$$N = \{y_0, y_1, \ldots, y_n\} \in \langle Y \rangle$$

and for any

$$\{y_{i_0}, y_{i_1}, \ldots y_{i_k}\} \in N \cap D,\ \varphi_N(\Delta_K) \subset D$$

where

$$\Delta_K = co\{e_{i_j} : j = 0,1,\cdots,k\})$$

It is easy to see that each FC-subspace of (Y, φ_N) is also an FC-space.

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Definition 2.2: Let X be a topological space and $(Y_i,\{\varphi_{N_i}\})$ be a family of FC-spaces where I is a finite or infinite index set and

$$Y=\prod_{i\in I}Y_i$$

The class B(Y, X) of better admissible mappings was introduced and defined as follows: $T\in B(Y,X)$ if and only if T: $Y\to 2^X$ is an upper semicontinuous set-valued mapping with compact values such that for each

$$i\in I, M_i\in<Y_i>(|M_i|=m_i+1)$$

and for any continuous mapping

$$\Psi:T(\prod_{i\in I}\varphi_{M_i}(\Delta_{m_i}))\to D$$

the composition mapping

$$\Psi\circ T|_{\prod_{i\in I}\varphi_{M_i}(\Delta_{m_i})}\circ\Phi:D\to 2^D$$

has a fixed point, where

$$D=\prod_{i\in I}\Delta_{m_i}$$

$$\Phi(t)=\prod_{i\in I}\varphi_{M_i}(\pi_i(t))$$

for all $t\in D$ and π_i is the projection of $D\to\Delta_{m_i}$.

When (Y, φ_N) is a G-convex space, I is a singleton, the notion of the class of better admissible mappings coincides with the corresponding notion introduced by Park [6]. The class B(Y, X) of better admissible set-valued mappings includes many important classes of set-valued mappings, for example, $U_c^k(Y,X)$ in [7], KKM(Y,X) in [8] and A(Y,X) in [9] and so on as proper subclasses.

COINCIDENCE THEOREMS

Theorem 3.1: Let X be a topological space and $(Y_i,\{\varphi_{N_i}\})$ be a family of FC-spaces where I is a finite or infinite index set. Let

$$Y=\prod_{i\in I}Y_i$$

and $T\in B(Y,X)$. For each $i\in I$, Let $F_i,G_i:X\to 2^{Y_i}$ be two set-valued mappings such that for each $i\in I$, for each $x\in X$ and

$$N_i=\{y_{i0},y_{i1},\cdots y_{in_i}\}\in<F_i(x)>,\varphi_{N_i}(\Delta_{n_i})\subset G_i(x)$$

for each nonempty subset K of X,

$$K=\bigcup_{y_i\in Y_i}(\mathrm{cint}F_i^{-1}(y_i)\cap K)$$

there exists a nonempty subset Y_i^0 of Y_i such that for each $N_i\in Y_i>$, there is a compact FC-subspace L_{N_i} of Y_i containing $Y_i^0\cup N_i$ and the set

$$D_i=\bigcap_{y_i\in Y_i^0}(\mathrm{cint}F_i^{-1}(y_i))^c$$

is empty or compact in X, where $(\mathrm{cint}F_i^{-1}(y_i))^c$ denotes the complement of $(\mathrm{cint}F_i^{-1}(y_i))$ in X.

Then there exists $\hat{y}\in Y$ and $\hat{x}\in X$ such that for each $i\in I$, $\hat{x}\in T(\hat{y})$ and $\hat{y}_i\in G_i(\hat{x})$.

Proof: For each fixed $i\in I$, if

$$D_i=\bigcap_{y_i\in Y_i^0}(\mathrm{cint}F_i^{-1}(y_i))^c$$

is empty in X, then

$$X=X\setminus D_i=\bigcup_{y_i\in Y_i^0}\mathrm{cint}F^{-1}(y_i)\qquad(3.1)$$

If D_i is nonempty and compact, by condition (ii), we have

$$D_i=\bigcup_{y_i\in Y_i^0}(\mathrm{cint}F_i^{-1}(y_i)\cap D_i)\subset\bigcup_{y_i\in Y_i^0}\mathrm{cint}F_i^{-1}(y_i)$$

Since D_i is compact, there exists a finite subset

$$N=\{y_{i0},y_{i1},\dots y_{in_i}\}\in<Y_i>$$

such that

$$D_i=\bigcap_{y_i\in Y_i^0}(\mathrm{cint}F_i^{-1}(y_i))^c\subset\bigcup_{k=0}^{n_i}\mathrm{cint}F_i^{-1}(y_{ik}).$$

It follows that

$$X=\bigcup_{y_i\in Y_i^0}\mathrm{cint}F^{-1}(y_i)\cup(\bigcup_{k=0}^{n_i}\mathrm{cint}F^{-1}(y_{ik}))\qquad(3.2)$$

Hence, in both cases that D_i is empty or nonempty compact, (3.2) always holds. By condition (iii), there exists a compact FC-subspace L_{N_i} of Y_i such that $Y_i^0\cup N_i\subset L_{N_i}$. By (3.2), we obtain

$$X\subset\bigcup_{y_i\in L_{N_i}}\mathrm{cint}F^{-1}(y_i)\qquad(3.3)$$

Let

$$L_N = \prod_{i\in I} L_{N_i}$$

Then L_N is a compact subset of Y. Since T is upper semicontinuous with compact values, by Proposition 3.1.11 of Aubin and Ekeland [10], $T(L_N)$ is a compact subset of X. By (3.3), we have that, for each $i\in I$,

$$T(L_N) \subset \bigcup_{y_i\in L_{N_i}} \mathrm{cint}F^{-1}(y_i)$$

Thus there exists

$$M_i = \{z_{i0}, z_{i1}, \ldots z_{im_i}\} \in \langle L_{N_i}\rangle$$

such that

$$T(L_N) = \bigcup_{i=0}^{m_i} (\mathrm{cint}F^{-1}(y_i) \cap T(L_N)) \qquad (3.4)$$

Since L_{N_i} is also an FC-subspace, there exists a continuous mapping $\varphi_{M_i} : \Delta_{m_i} \to Y$,such that for each $B_i \in \langle M_i\rangle$,

$$\varphi_{M_i}(\Delta_J) \subset B_i, |B_i| = |J+1| \qquad (3.5)$$

By (3.4), we may assume that $\{\psi_{i_k}\}_{k=0}^{m_i}$ is a continuous partition of unity subordinated to the open covering $\{\mathrm{cint}F^{-1}(y_i) \cap T(L_N)\}_{i=0}^{m_i}$ such that for each

$$k = 1,2,\cdots, m_i, \psi_{i_k} : T(L_N) \to [0,1]$$

is continuous,
for each $k = 1,2,\ldots,m_i$ and

$$x \in T(L_N), \psi_{i_k}(x) \neq 0 \Leftrightarrow x \in ����F^{-1}(z_{i_k}) \Rightarrow z_{i_k} \in F_i(x)$$

for each

$$x \in T(L_N), \sum_{k=0}^{m_i} \psi_{i_k}(x) = 1.$$

For each $i\in I$, define a mapping $\psi_i : T(L_N) \to \Delta_{m_i}$ as follows: for each $x\in T(L_N)$,

$$\psi_i(x) = \sum_{k=0}^{m_i} \psi_{i\,k}(x) e_{i\,k}$$

where

$$\{e_{i\,k} : k = 0,1,\cdots m_i\}$$

are the vertices of standard m_i-dimensional simplex Δm_i. Then ψ_i is continuous and for each

$$x \in T(L_N), \psi_i(x) = \sum_{k=0}^{m_i} \psi_{i_k}(x) e_{i_k}$$

where

$$J(x) = \{k \in \{0,1,\cdots,m_i\} : \psi_{i_k} \neq 0\}$$

By the property (2), we have

$$\{z_{i_k} : k \in J(x)\} \in \langle F_i(x)\rangle$$

By (3.5) and condition (i), we obtain that for each $x\in T(L_N)$,

$$\phi_{M_i} \circ \varphi_i(x) \in \phi_{M_i}(\Delta_{J(x)}) \subset \varphi_{N_i}(\{z_{i_k} : k \in J(x)\}) \subset G_i(x) \quad (3.6)$$

Let

$$D = \prod_{i\in I} \Delta m_i$$

define continuous mappings ΦL $D \to L_N$ and $\Psi : T(L_N) \to D$ as follows that for each

$$t \in D, \Phi(t) = \prod_{i\in I} \varphi_{M_i}(\pi_i(t))$$

and for each

$$x \in T(L_N), \Psi(x) = \prod_{i\in I} \psi_i(x)$$

where $\pi_i : D \to \Delta_{m_i}$ is projection of D onto Δ_{m_i}. Note that for each $i \in I, M_i \subset L_{N_i}$ and L_{N_i} is FC-subspace, we have $\varphi_{N_i}(\Delta_{m_i}) \subset L_{N_i}$ and hence

$$\prod_{i\in I} \varphi_{N_i}(\Delta_{m_i}) \subset \prod_{i\in I} L_{N_i} = L_N$$

and

$$T(\prod_{i\in I} \varphi_{N_i}(\Delta_{m_i})) \subset T(L_N)$$

Then we have

$$\Psi \circ T|_{\prod_{i\in I}\varphi_{N_i}(\Delta_{m_i})} \circ \Phi : D \to 2^D$$

Since $T\in B(Y,X)$, there exists a point $t\in D$ such that

$$t \in \Psi \circ T|_{\prod_{i\in I}\varphi_{N_i}(\Delta_{m_i})} \circ \Phi(t)$$

Letting $\hat{y} = \Phi(t)$, then there exists $\hat{x} \in T(\hat{y})$ such that

$$\hat{y} = \Phi \circ \Psi(\hat{x}) = \Phi(\prod_{i\in I} \psi_i(\hat{x})) = \prod_{i\in I} \varphi_{M_i} \circ \psi_i(\hat{x})$$

It follows from (3.6) that for each

$$i\in I,\ \hat{y}_i = \varphi_{M_i} \circ \psi_i(\hat{x}) \in G_i(\hat{x})$$

This completes the proof.

Remark 3.1: The condition (i) of Theorem 3.1 can be replaced by the following condition

(i) for each $x\in X$, $G_i(x)$ is a FC-subspace of Y.

Theorem 3.2: Let X be a topological space and $(Y_i,\{\varphi_{N_i}\})$ be a family of FC-spaces where I is a finite or infinite index set. Let

$$Y=\prod_{i\in I}Y_i$$

For each $i\in I$, Let $F_i,G_i: X\to 2^{Y_i}$ be two set-valued mappings such that for each $i\in I$,

(i) for each $x\in X$ and

$$N_i\in\langle F_i(x)\rangle,\varphi_{N_i}(\Delta_{n_i})\subset G_i(x)$$

(ii) for each nonempty subset K of X,

$$K=\bigcup_{y_i\in Y_i}(\mathrm{cint}F_i^{-1}(y_i)\cap K)$$

(iii) There exists a nonempty subset Y_i^0 of Y_i such that for each $N_i\in\langle Y_i\rangle$, there is a compact FC-subspace L_{N_i} of Y_i containing $Y_i^0\cup N_i$ and the set

$$D_i=\bigcap_{y_i\in Y_i^0}(\mathrm{cint}F_i^{-1}(y_i))^c$$

is empty or compact in X, where $(\mathrm{cint}F_i^{-1}(y_i))^c$ denotes the complement of $(\mathrm{cint}F_i^{-1}(y_i))$ in X.

Then for each continuous single-valued mapping $T:Y\to X$, there exists $\hat{y}\in Y$ such that $\hat{y}_i\in G_i(T(\hat{y}))$ for each $i\in I$.

Proof: By using same argument as in the proof of Theorem 3.1, we can get that

$$\Psi\circ T|_{\prod_{i\in I}\varphi_{N_i}(\Delta_{m_i})}\circ\Phi: D\to D$$

is a continuous single-valued mapping. By Tychonoff's fixed point theorem, there exists a point $t\in D$ such that

$$t\in\Psi\circ T|_{\prod_{i\in I}\varphi_{N_i}(\Delta_{m_i})}\circ\Phi(t)$$

Letting $\hat{y}=\Phi(t)$, then we have

$$\begin{aligned}\hat{y}&=\Phi\circ\Psi(T(\hat{y}))=\Phi(\prod_{i\in I}\psi_i(T(\hat{y})))\\&=\prod_{i\in I}\varphi_{M_i}\circ\psi_i(T(\hat{y}))\end{aligned}.$$

It follows from (3.6) that for each

$$i\in I,\hat{y}_i=\varphi_{M_i}\circ\psi_i(\hat{x})\in G_i(T(\hat{y}))$$

Corollary 3.1: Let X be a topological space and $(Y_i,\{\varphi_{N_i}\})$ be a family of FC-spaces where I is a finite or infinite index set. Let

$$Y=\prod_{i\in I}Y_i$$

For each $i\in I$, Let $F_i,G_i: X\to 2^{Y_i}$ be two set-valued mappings such that the condition (i) and (ii) in Theorem 3.2 are satisfied. Then for any continuous single-valued mapping $T:Y\to X$, there exists $\hat{y}\in Y$ such that $\hat{y}_i\in G_i(T(\hat{y}))$ for each $i\in I$.

Proof: Since for each $i\in I$, Y_i is a compact FC-subspace, by letting $Y_i^0=L_{N_i}=Y_i$, for each $N_i\in\langle Y_i\rangle$ and $i\in I$, then it follows from condition (ii) that

$$X=\bigcup_{y_i\in Y_i}\mathrm{cint}F_i^{-1}(y_i)$$

and hence for each $i\in I$,

$$\begin{aligned}D_i&=\bigcap_{y_i\in Y_i^0}(\mathrm{cint}F_i^{-1}(y_i))^c=\bigcap_{y_i\in Y_i}(\mathrm{cint}F_i^{-1}(y_i))^c\\&=X\setminus\bigcap_{y_i\in Y_i}\mathrm{cint}F_i^{-1}(y_i)=\Phi\end{aligned}.$$

The conclusion holds by Theorem 3.2.

Remark 3.2: Theorem 3.1, Theorem 3.2 and Corollary 3.1 generalize Theorem 3.1, Theorem 3.2 and Corollary 3.1 in Ding [3] from G-convex space to FC-space.

When I is a singleton, from Theorem 3.1, we can obtain the following result.

Corollary 3.2: Let X be a topological space and (Y,φ_N) be an FC-space. Let $T\in B(Y,X)$ and $F,G:X\to 2^Y$ be set-valued mappings such that for each $x\in X$ and

$$N\in\langle F(x)\rangle,\varphi_N(\Delta_n)\subset G(x)$$

for each nonempty subset K of

$$X,\ K=\bigcup_{y\in Y}(\mathrm{cint}F^{-1}(y)\cap K)$$

there exists a nonempty subset Y^0 of Y such that for each $N \in \langle Y \rangle$, there is a compact FC-subspace L_N of Y containing $Y^0 \cup N$ and the set

$$D = \bigcap_{y \in Y^0} (\text{cint}F^{-1}(y))^c$$

is empty or compact in X, where $(\text{cint}F^{-1}(y))^c$ denotes the complement of $(\text{cint}F^{-1}(y))$ in X.

Then there exists $\hat{x} \in X$ and $\hat{y} \in Y$ such that $\hat{x} \in T(\hat{y})$? $\hat{y} \in G(\hat{x})$.

Remark 3.3: Corollary 3.2 generalizes Corollary 3.2 in Ding [3] from G-convex space to FC-space.

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