

Laplace Adomian Decomposition Method for Solving the Nonlinear Volterra Integral Equation with Weakly Kernels

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Abstract: A computational method (The Combined Laplace Adomian decomposition method) applied for solving linear and nonlinear Volterra integral equation with weakly kernels, we discuss some examples, when the kernel takes a Carleman and logarithmic forms to demonstrate the high accuracy and simplicity of our method. A comparison with the Toeplitz matrix method is made.

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INTRODUCTION

In practical applications one frequently encounters the Volterra integral equations of the second kind with a weakly singular kernel of the form

$$y(x) = g(x) + \lambda \int_a^x p(x,t)k(x,y(t))dt \quad (1)$$

where the kernel $p(x,t)$ is weakly singular and the given functions $g(x)$ and $k(x,y(t))$ are assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution $y \in C[a,x]$, [1-3]. Typical forms of $p(x,t)$ are

- (i) $p(x,t) = |x-t|^{-\alpha}$, $0 < \alpha < 1$
- (ii) $p(x,t) = \ln|x-t|$

For volterra equations with bounded kernels, the smoothness of the kernel and of the forcing function $g(x)$ determines the smoothness of the solution on the closed interval $[a,x]$ with $x>a$. If we allow weakly singular kernels, then the resulting solutions are typically non-smooth at the initial point of the interval of integration, where their derivatives become unbounded. Some results concerning the behavior of the exact solutions of equations of type (1) are given in [4, 5].

It is well known that the nonlinear Volterra integral equation (1) is usually handled by many methods, such as Toeplitz matrix method [6] and others. The traditional Laplace method by itself cannot be used in this case because of the nonlinearity of this equation, then from [7] there exists a method combining the Laplace transform method with Adomian decomposition method for analytic treatment of nonlinear singular integral equation that describe heat transfer. If $p(x,t)$ of(1) is a difference kernel $p(x-t)$. The nonlinear Volterra integral equation (1) can be expressed as:

$$y(x) = g(x) + \lambda \int_0^x p(x-t)k(y(t))dt \quad (2)$$

so it is possible to overcome this difficulty of nonlinearity by the powerful Adomian decomposition method.

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In this paper we consider the nonlinear Volterra integral equations with weakly kernels of the forms (i) and (ii). The combined algorithm is capable of handling the two types singular Volterra integral equations, then we compare the results with the numerical solution for the singular Volterra integral equations by using Toeplitz matrix method [6].

CARLEMAN KERNEL

We assume that the kernel $p(x,t)$ of (1) takes the form (i), then nonlinear Volterra integral equation(1) can be expressed as:

$$y(x) = g(x) + \lambda \int_0^x |x-t|^{-\alpha} k(y(t)) dt \quad (3)$$

Applying the Laplace transform to both sides of (3) gives:

$$Y(s) = G(s) + \lambda L\{|x-t|^{-\alpha}\} L\{k(y(t))\} \quad (4)$$

The Adomian decomposition method can be used to handle (4). We represent the linear term $Y(s)$ by

$$Y(s) = \sum_{n=0}^{\infty} Y_n(s) \quad (5)$$

So

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \quad (6)$$

The nonlinear term $k(y(t))$ will be represented by the Adomian polynomials A_n in the form:

$$k(y(x)) = \sum_{n=0}^{\infty} A_n(x) \quad (7)$$

where A_n , $n \geq 0$ are given by

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} k \left(\sum_{i=0}^n \lambda^i y_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (8)$$

where the so called Adomian polynomials A_n can be evaluated for all forms of nonlinearity. Substituting (5) and (6) into (4) leads to

$$\sum_{n=0}^{\infty} Y_n = G(s) + \lambda L\{|x-t|^{-\alpha}\} L\left\{ \sum_{n=0}^{\infty} A_n(x) \right\} \quad (9)$$

The Adomian decomposition method introduces the recursive relation

$$Y_0(s) = G(s), \quad Y_{k+1}(s) = \lambda \left[s^{\alpha-1} \sqrt{1-\alpha} \right] L\{A_k(x)\}, \quad k \geq 0 \quad (10)$$

Appling the inverse Laplace transform to the first part of (9) gives $y_0(x)$, that will define A_0 . Using $A_0(x)$ will enable us to evaluate $y_1(x)$. $A_k(x)$ are the Adomian for the nonlinear term. The Adomian decomposition method assumes that the linear term $y(x)$ can be decomposed by the series. The obtained series solution may converge to an exact solution if such a solution exists.

Example 1 (Linear case) [6]: Consider the nonlinear volterra integral equation with Carleman kernel

Table 1:

x	y _E	y _{LA}	E _{LA}	y _{TM}	E _{TM}
0.70	4.9000E-01	4.94310E-01	4.3103E-03	4.88070E-01	1.9250E-03
0.56	3.1360E-01	3.15815E-01	2.8148E-03	3.12548E-01	1.0518E-03
0.42	1.7640E-01	1.77352E-01	1.3518E-04	1.75918E-01	4.8190E-04
0.28	7.8400E-02	7.86961E-02	2.9608E-04	7.82400E-02	1.6004E-04
0.14	1.9600E-02	1.96421E-02	4.2108E-05	1.95760E-02	2.4033E-05
0.10	1.0000E-02	1.00166E-02	1.6580E-05	4.98990E-03	1.0059E-05
0.08	6.4000E-03	6.40898E-03	8.9790E-06	6.39450E-03	5.4900E-06
0.06	3.6000E-03	3.60409E-03	4.0859E-06	3.59740E-03	2.5180E-06
0.04	1.6000E-03	1.60136E-03	1.3551E-07	1.59916E-03	8.3620E-07
0.02	4.0000E-04	4.00208E-04	2.0780E-07	3.99870E-04	1.2550E-07

$$y(x) = x^2 + \lambda \int_0^x |x-t|^{-\alpha} k(y(t)) dt \quad (11)$$

where the exact solution is $y(x) = x^2$, $\alpha = 1/3$, $\lambda = 0.01$. By taking Laplace transform we obtain

$$Y(s) = \frac{2}{s^3} + \lambda \frac{\Gamma[\frac{2}{3}]}{s^{\frac{2}{3}}} Y[s] \quad (12)$$

The Adomian decomposition method introduces the recursive relation

$$Y_0(s) = \frac{2}{s^3}, \quad Y_{k+1}(s) = \lambda \left[\frac{\Gamma[\frac{2}{3}]}{s^{\frac{2}{3}}} \right] Y_k(s), \quad k \geq 0 \quad (13)$$

so that, taking the inverse Laplace transform of both sides of relation (13) gives

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad y_0(x) = x, \quad y_1(x) = \lambda \left[\frac{x^{\frac{8}{3}} \Gamma[\frac{2}{3}]}{\Gamma[\frac{11}{3}]} \right]$$

$$y_2(x) = 2\lambda \left[\frac{x^{\frac{10}{3}} (\Gamma[\frac{2}{3}])^2}{\Gamma[\frac{13}{3}]} \right], \quad y_3(x) = \frac{\lambda}{12} [x^4 (\Gamma[\frac{2}{3}])^3], \quad y \equiv y_0 + y_1 + y_2$$

We denote by y_E to the exact solution and y_{LA} , y_{TM} to Laplace Adomian and Toeplitz matrix solutions. A comparison of the results shown in Table 1.

Example 2 (Nonlinear case) [8]

$$y(x) = x^{\frac{1}{2}} (1 - \frac{4}{3}x) + \int_0^x (x-t)^{-\frac{1}{2}} y^2(t) dt \quad (14)$$

where the exact solution $y(x) = x^{1/2}$. By taking Laplace transform we obtain

$$Y(s) = \frac{\sqrt{\pi}(-2+s)}{2s^{\frac{5}{2}}} + \frac{\sqrt{\pi}}{s} L \{ y^2 t \} \quad (15)$$

Table 2:

x	y _E	y _{LA}	E _{LA}
0.5	0.0707071	0.4221380	2.84969E-01
0.4	0.6324560	0.4779700	1.54480E-01
0.3	0.5477230	0.4221378	6.61484E-02
0.2	0.4472140	0.4286330	1.85808E-02
0.1	0.3162280	0.3143370	1.89004E-03
0.08	0.2828430	0.2819530	8.89551E-04
0.06	0.2449490	0.2446150	3.33846E-04
0.04	0.2000000	0.1999170	8.29380E-05
0.02	0.1414210	0.1414141	7.52568E-06
0.01	0.1000000	0.0999993	6.73892E-07

The Adomian decomposition method admits the use of

$$Y_0(s) = \frac{\sqrt{\pi}(-2+s)}{2s^{\frac{5}{2}}}, Y_{k+1}(s) = \frac{\sqrt{\pi}}{s} L \{ A_k(x) \}, k \geq 0 \quad (16)$$

where the Adomian polynomials are:

$$A_0(x) = y_0^2, A_1(x) = 2y_0y_1, A_2(x) = \frac{1}{2}(2y_1^2 + 4y_0y_2) \quad (17)$$

and so on for other Adomian polynomials. Using the recurrence relation (7), we find

$$\begin{aligned} y_0(x) &= x^{\frac{1}{2}}(1 - \frac{4}{3}x), y_1(x) = \frac{4}{315}x^{\frac{3}{2}}(105 - 224x + 128x^2) \\ y_2(x) &= \frac{-128x^{\frac{5}{2}}}{3274425}(-72765 + 21621x - 225280x^2 + 81920x^3), y \equiv y_0 + y_1 + y_2 \end{aligned} \quad (18)$$

Comparing the results with the exact solution given in Table 2.

LOGARITHMIC KERNEL

If the kernel p(x,t) of (1) takes the form (ii), then nonlinear Volterra integral equation(1) can be expressed as :

$$y(x) = g(x) + \lambda \int_0^x \ln|x-t| k(y(t)) dt \quad (19)$$

Applying the Laplace transform to both sides of (19) gives:

$$Y(s) = G(s) + \lambda \left(\frac{-\gamma + \ln(s)}{s} \right) L[k(y(t))] \quad (20)$$

where γ is Euler's constant. Using the same method we shall find at the end the required solution by the inverse of Laplace transform.

Example 3 (Linear case) [6]

$$y(x) = x^2 + \lambda \int_0^x \ln(t-x) y(t) dt \quad 0 < t < \infty \quad (21)$$

Table 3:

x	y _E	y _{LA}	E _{LA}	y _{TM}	E _{TM}
0.70	4.90000E-01	4.88643E-01	1.35690E-03	4.96324E-01	6.324070E-03
0.56	3.13600E-01	3.12750E-01	8.49510E-04	3.16839E-01	3.239250E-03
0.42	1.76400E-01	1.75953E-01	4.46900E-04	1.77768E-01	1.368100E-03
0.14	1.96000E-02	1.95704E-02	2.95730E-05	1.96507E-02	5.071660E-05
0.10	1.00000E-02	9.98780E-02	1.22018E-05	1.00181E-02	1.813730E-05
0.08	6.40000E-03	6.39328E-03	6.72190E-06	6.40928E-03	9.277800E-06
0.06	3.60000E-03	3.59691E-03	3.08908E-06	3.60391E-03	3.905910E-06
0.04	1.60000E-03	1.59898E-03	1.01830E-06	1.60115E-03	1.150410E-06
0.02	4.00000E-04	3.99852E-04	1.48446E-07	4.00140E-04	1.397160E-07

where the exact solution $y(x) = x^2$, $\lambda = 0.01$. By taking Laplace transform, we obtain

$$Y(s) = \frac{2}{s^3} + \lambda \left(\frac{-\gamma + \ln(s)}{s} \right) L\{y(t)\} \quad (22)$$

The Adomian decomposition method introduces the recursive relation

$$Y_0(s) = \frac{2}{s^3}, Y_{k+1}(s) = \lambda \left[\frac{-\gamma + \ln s}{s} \right] Y_k(s), k \geq 0 \quad (23)$$

so that

$$\begin{aligned} y_0(x) &= x^2, y_1(x) = \frac{\lambda}{18} x^3 [-11x + 61\ln x], \\ y_2(x) &= \frac{\lambda}{864} x^4 [415 - 12\pi^2 - 300\ln x + 72(\ln x)^2], y \equiv y_0 + y_1 + y_2 \end{aligned}$$

Comparing this results by the Toeplitz matrix method given in Table 3.

Example 4 (Nonlinear case)[6]

$$y(x) = x^2 + \lambda \int_0^x \ln(x-t) y(t) dt \quad (24)$$

where the exact solution $y(x) = x^2$, $\lambda = 7.70691506$. By taking Laplace transform, we obtain

$$Y(s) = \frac{2}{s^3} + \lambda \left(\frac{-\gamma + \ln(s)}{s} \right) L\{y(t)\} \quad (25)$$

The Adomian decomposition method introduces the recursive relation

$$Y_0(s) = \frac{2}{s^3}, Y_{k+1}(s) = \lambda \left[\frac{-\gamma + \ln s}{s} \right] Y_k(s), k \geq 0 \quad (26)$$

so that

$$\begin{aligned} y_0(x) &= x^2, y_1(x) = \frac{-1}{980} x^7 [363 - 140 \ln x] \\ y_2(x) &= \frac{1}{19756800} x^8 [3144919 - 58800\pi^2 - 1917720 \ln x + 352800(\ln x)^2], y \equiv y_0 + \lambda(y_1 + y_2) \end{aligned}$$

Results in Table 4.

Table 4:

x	y_E	y_{LA}	E_{LA}	y_{TM}	E_{TM}
0.10	1.00000E-02	0.99995E-02	5.04456E-07	9.99999E-03	1.20030E-08
0.08	6.40000E-03	6.39989E-03	1.11864E-07	6.40000E-03	2.65600E-09
0.06	3.60000E-03	3.59998E-03	1.59580E-08	3.60000E-03	2.90000E-10
0.04	1.60000E-03	1.60000E-03	1.01667E-09	1.60000E-03	9.00000E-12
0.02	4.00000E-04	4.00000E-04	9.01265E-12	4.00000E-04	4.00000E-13

CONCLUSION

In this work we showed the accuracy and simplicity of Combined Laplace-Adomian method applied to linear and nonlinear Volterra integral equations with weakly kernels. This method presents a useful way to develop computational method for solving these kinds of nonlinear singular integral equations. We made a comparison with Toeplitz matrix method for solving the same problem.

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