Studies in Nonlinear Sciences 2 (4): 123-128, 2011
ISSN 2221-3910
© IDOSI Publications, 2011

Exact Travelling Wave Solution of Nonlinear Physical Models

[1,2]A.A. Soliman and [3,4]M.A. Abdou

[1]Department of Mathematics,
Faculty of Education (AL-Arish) Suez Canal University, AL-Arish, 45111, Egypt
[2]Department of Mathematics, Bisha Faculty of science & Arts,
King Khalid University, Bisha, PO Box 344, Saudi Arabia
[3]Theoretical Research Group, Physics Department,
Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
[4]Department of Physics, Faculty of Education for Girls,
Science Departments, King Khalid University, Bisha, Saudi Arabia

Abstract: This paper suggested a new method for constricting a new exact travelling wave solution nonlinear physical equations. A transformation and a general function for travelling wave are introduced, using the Maple, a general exact solution can be readily obtained. By suitable choice of the parameters, the obtained solution reduces to various solitary and periodic solutions. The generalized regularized long-wave equation, Camassa-Holm equation, nonlinear Fokker-Planck equation, generalized Burgers-Fisher equation are used as illustrative examples to show the effectiveness and convenience of the method. The proposed method can be used to solve another nonlinear physics problems.

Key words: Nonlinear physical equations . solitary and periodic solutions

INTRODUCTION

As more and more problems in branches of modern mathematics, physics and other interdisciplinary science are described in terms of suitable nonlinear models, directly exploring explicit and exact solutions to nonlinear evolution equations plays a very important role in nonlinear science, particularly in nonlinear physics science. In the past several decades, both mathematicians and physicists have made significant progress in this direction.

Many effective methods [1-13] have been presented such as variational iteration method [6], homotopy perturbation method [3], Exp-function method [13] and others. A complete review on the field is available on [3].

In this paper, by introducing a transformation and selecting appropriate, we successfully find rich explicit and exact solutions of four famous and physically important nonlinear evolution equations namely, the generalized regularized long-wave (RLW) equation, the Camassa-Holm equation, nonlinear Fokker Planck (FP) equation and the generalized Burgers-Fisher equation

The rest of this paper is arranged as follows. In Section 2, we briefly describe the proposed method. In Section 3, as illustrative examples, we apply it to solving four physically significant nonlinear evolution equations. Some conclusion are given in the last section.

METHODOLOGY

For a given the nonlinear evolution equations, say, in two independent variables x and t

$$Q\left(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 u}{\partial x^2},.....\right)=0 \qquad (1)$$

The left hand side of Eq.(1) is a polynomial in terms of u and its various partial derivatives. In order to solve Eq.(1), we use the transformation

$$u=u_0+\frac{\partial v}{\partial x}, v=\frac{\partial \ln w}{\partial x}, w=w(y), y=y(x,t) \qquad (2)$$

in which w(y) and y(x,t) are two functions.

It is known that Eq.(1) are nonlinear equation and that their solutions should contain the phase fracture (kx-wt).As a result, we straightforwardly choose the trial function y(x,t) in the following form

Corresponding Author: A.A. Soliman, Department of Mathematics, Faculty of Education (AL-Arish) Suez Canal University, AL-Arish, 45111, Egypt

$$y = e^{(kx-wt)} \quad (3)$$

where k and w are the wave number and the angular frequency, respectively.
We Suggest another function w(y) as:

$$w(y) = [a + y^2]^n \quad (4)$$

where a and n are constants to be determined later.

In view of Eq.(2) along with Eq.(3) as well as Eq.(4), we have

$$v = \frac{\partial \ln w}{\partial x} = \frac{2nky^2}{[a + y^2]} \quad (5)$$

$$u = u_0 + \frac{\partial v}{\partial x} = u_0 + \frac{4ank^2 y^2}{[a + y^2]^2} \quad (6)$$

In what follows, let us describe the proposed method as follows. Firstly, substituting the above equations into Eq.(1) yields a set of algebraic equations with regard to the unknown variables n and a and equating the coefficients of all power of y to zero. Secondly, by solving the obtained system of algebraic equation for n and a with the aid of Maple, we obtain the unknown constants n and a. Lastly, inserting n into Eq.(6) and using Eq.(3) along with choosing a = 1 and a = -1, respectively, then many new exact travelling wave solution are obtained. In the following, we shall apply the technique stated above to solve four physically important nonlinear evolution equations as illustrative examples.

NEW APPLICATIONS

The generalized regularized long-wave (RLW) equation: Let us first consider the generalized regularized long-wave equation [14] reads

$$u_t + u_x + \alpha(u^p)_x - \beta u_{xxt} = 0 \quad (7)$$

where p is a positive integer and α and β are positive constants. Eq.(7) was first put forward as a model for small amplitude long waves on the surface of water in a channel by Peregine [15, 16] and later by Benjamin [17]. In physical situation such as unidirectional waves propagating in a water channel, long crested waves in near shore zones and many other, the generalized long wave (RLW) equation serves an alternative model to the KdV equation [18].

Our main goal is to solve Eq.(7) by means of the proposed method illustrated above. Substituting Eqs. (3-6) into Eq. (7) and collecting the coefficients of powers of y with the aid of the computerized symbolic computation of the powerful Maple, then setting each of the obtained coefficients to zero, give rise to a set of algebraic equations with respect to the unknown variables a and n as follows:

$$\begin{aligned} &-8nk^2a[wa^2 - a^2k - 2\alpha k u_0 a^2 - 8\alpha n a^2 k^3 + 44\beta w a^2 k^2] = 0 \\ &-8nk^2a[-wa + ka + 2\alpha k u_0 a + 8\alpha n a k^3 - 44\beta w a k^2] = 0 \\ &-8nk^2a[-w + k + 2\alpha k u_0 + 4\beta w k^2] = 0 \end{aligned} \quad (8)$$

In view of Eqs.(8), we have

$$a = a, n = \frac{6w\beta}{\alpha k}, u_0 = -\frac{1}{2}\frac{-w + k + 4\beta wk^2}{k\alpha} \quad (9)$$

$$c = \frac{w}{k} \quad (10)$$

Inserting Eq.(9) into Eq.(6) and taking into account Eq.(3) and Eq.(10) simultaneously, we obtain the general travelling wave solution the regularized long-wave equation (7) as follows

$$u(x,t) = -\frac{1}{2}\frac{-w + k + 4\beta wk^2}{k\alpha} + \frac{24kw\beta a e^{2k(x-ct)}}{\alpha(a + e^{2k(x-ct)})^2} \quad (11)$$

where three arbitrary constants a, w and k.
Making use of the following identity

$$\frac{e^x}{e^{2x}+1} = \frac{1}{2}\operatorname{sech}x \quad (12)$$

and setting a = 1 in Eq.(11),we get the so-called bell-type solitary wave solution to the regularized long-wave Eq.(7) in the following form

$$u(x,t) = -\frac{1}{2}\frac{-w + k + 4\beta wk^2}{k\alpha} + \frac{6w\beta k}{\alpha}\operatorname{sech}^2 k(x - ct) \quad (13)$$

Making use of the following identity

$$\frac{e^x}{e^{2x}-1} = \frac{1}{2}\operatorname{csch}x \quad (14)$$

and setting a = -1 in Eq.(11), we have the singular travelling wave solution to regularized long-wave equation (7) as follows

$$u(x,t) = -\frac{1}{2}\frac{-w + k + 4\beta wk^2}{k\alpha} - \frac{6w\beta k}{\alpha}\operatorname{csch}^2 k(x - ct) \quad (15)$$

Making use of the following identity

$$\operatorname{sech}^2 x = \frac{2}{\cosh 2x + 1} \tag{16}$$

then Eq.(13) admits to the following

$$u(x,t) = -\frac{1}{2}\frac{-w + k + 4\beta wk^2}{k\alpha} + \frac{12wk\beta}{\alpha}\frac{1}{\cosh 2k(x-ct)+1} \tag{17}$$

Making use of the following identity

$$\operatorname{csch}^2 x = \frac{2}{\cosh 2x - 1} \tag{18}$$

then Eq.(15) can be converted to

$$u(x,t) = -\frac{1}{2}\frac{-w + k + 4\beta wk^2}{k\alpha} - \frac{12wk\beta}{\alpha}\frac{1}{\cosh 2k(x-ct)-1} \tag{19}$$

Let k = iK, where i is the imaginary unit and K the constant and making use of the following identity

$$\operatorname{sech}(ix) = \sec(x), \operatorname{csch}(ix) = -i\csc(x) \tag{20}$$

then Eq.(8) and Eq.(15) can be reduced to

$$u(x,t) = -\frac{1}{2}\frac{-w + iK - 4\beta wK^2}{iK\alpha} + \frac{12iKw\beta}{\alpha}\sec^2 K(x-ct) \tag{21}$$

$$u(x,t) = -\frac{1}{2}\frac{-w + iK - 4\beta wK^2}{iK\alpha} + \frac{12iKw\beta}{\alpha}\csc^2 K(x-ct) \tag{22}$$

which are two triangle function-type periodic wave solutions of the regularized long-wave (RLW) equation (7).

Camassa-Holm equation: A second interactive model is the Camassa-Holm equation [19] reads

$$u_t + 2pu_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{23}$$

Eq.(23) appeared first in a physical context as describing the shallow water approximation in invisible hydrodynamics [20, 21]. The variable u(x,t) represents the fluid velocity in the horizontal direction x and p is a constant.

In the same manner, to solve Eq. (23) using the proposed method. Similarly, putting Eqs. (3-6) into Eq. (23) and collecting the coefficients of powers of y with the aid of the computerized symbolic computation of the powerful Maple, then setting each of the obtained coefficients to zero, results in a system of over-determined algebraic equations with regard to the unknown variables a and n as:

$$\begin{aligned}
&-8nk^2a[-w + 2pk - 4k^3u_0 + 4wk^2 + 3ku_0] = 0\\
&-8nk^2aa[36au_0k^3 + 6pka - 36wak^2 + 12nak^3\\
&+9nak^3 + 9kau_0 - 48nak^5 - 3wa] = 0\\
&-8nk^2a[40a^2u_0k^3 + 4pa^2k - 40k^2wa^2 + 12a^2nk^3\\
&+6ku_0a^2 + 336na^2k^5 - 2wa^2] = 0
\end{aligned} \tag{24}$$

Solving the system of algebraic equations obtained above using Maple, we have

$$\begin{aligned}
&a = a, n = \frac{8(w+pk)}{k(-16k^2+3+16k^4)},\\
&u_0 = \frac{-w+2pk+4wk^2}{k(4k^2-3)}, c = \frac{w}{k}
\end{aligned} \tag{25}$$

Inserting Eq.(25) into Eq.(6) and making use of Eq.(3), admits the following the general travelling wave solution to the Camassa-Holm equation (23)

$$\begin{aligned}
u(x,t) = &\frac{-w+2pk+4wk^2}{k(4k^2-3)}\\
&+\frac{32(w+pk)kae^{2k(x-ct)}}{(-16k^2+3+16k^4)(a+e^{2k(x-ct)})^2}
\end{aligned} \tag{26}$$

With the aid of (12) and setting a = 1 in Eq. (26), we obtain the so-called bell type solitary wave solution to the Camassa-Holm equation (23) as follows

$$\begin{aligned}
u(x,t) = &\frac{-w+2pk+4wk^2}{k(4k^2-3)}\\
&+\frac{8(w+pk)k}{(-16k^2+3+16k^4)}\operatorname{sech}^2 k(x-ct)
\end{aligned} \tag{27}$$

By means of Eq. (14) and setting a = -1 in Eq.(26), we possess the singular travelling wave solution of the Camassa-Holm equation (23) as follows

$$\begin{aligned}
u(x,t) = &\frac{-w+2pk+4wk^2}{k(4k^2-3)}\\
&-\frac{8(w+pk)k}{(-16k^2+3+16k^4)}\operatorname{csch}^2 k(x-ct)
\end{aligned} \tag{28}$$

Similarly, making use of the former two equalities (16) and (18), admits Eq.(27) and Eq.(29) to the following form

$$\begin{aligned}
u(x,t) = &\frac{-w+2pk+4wk^2}{k(4k^2-3)}\\
&+\frac{16(w+pk)k}{(-16k^2+3+16k^4)}\frac{1}{\cosh 2k(x-ct)+1}
\end{aligned} \tag{29}$$

$$u(x,t)=\frac{-w+2pk+4wk^2}{k(4k^2-3)}-\frac{16(w+pk)k}{(-16k^2+3+16k^4)}\frac{1}{\cosh 2k(x-ct)-1} \quad (30)$$

Similarly, making use of the previous identity (20),then Eq.(27) and Eq.(28) can be reduced to

$$u(x,t)=\frac{-w+2ipK-4wK^2}{iK(-4K^2-3)}+\frac{8(w+ipK)iK}{(16K^2+3+16K^4)}\sec^2 K(x-ct) \quad (31)$$

$$u(x,t)=\frac{-w+2ipK-4wK^2}{iK(-4K^2-3)}-\frac{8(w+ipK)iK}{(16K^2+3+16K^4)}\csc^2 K(x-ct) \quad (32)$$

which are two triangle function-type periodic wave solutions of the Camassa-Holm equation (23).

The nonlinear Fokker Planck (FP) equation: Consider the nonlinear Fokker-Planck (FP) equation [22] in the form

$$u_t=u_x+\lambda(u^2)_{xx}+Du_{xx} \quad (33)$$

where D is the diffusion coefficient and λ is constant. The nonlinear Fokker-Planck (FP) equation (33) which appears for the macroscopic of generalized Langevin equations [23, 24].

Inserting Eqs. (3-6) into Eq.(33) and collecting the coefficients of powers of y with the aid of the computerized symbolic computation of the powerful Maple, then setting each of the obtained coefficients to zero, results in a system of over-determined algebraic equations with regard to the unknown variables a and n as follows:

$$\begin{aligned}
&-8nak^2[wa^2+ka^2+2ka^2u_0+8na^2k^3-6Dk^2a^2]=0\\
&-8nak^2[-wa-ka-2kau_0-8nak^3-6Dk^2a]=0\\
&-8nak^2[-w-k-2ku_0+2Dk^2]=0
\end{aligned} \quad (34)$$

Solving the system of algebraic equations obtained above, we obtain the following solutions

$$a=a,\ n=-\frac{D}{k},\ u_0=\frac{1}{2}\frac{-w-k+2Dk^2}{k} \quad (35)$$

Substituting Eq.(35) into Eq.(6) and taking into consideration Eq. (3) and Eq. (10) simultaneously, we obtain the following the exact travelling wave solution for the nonlinear Fokker-Planck equation (33)

$$u(x,t)=\frac{1}{2}\frac{-w-k+2Dk^2}{k}-\frac{4kDae^{2k(x-ct)}}{(a+e^{2k(x-ct)})^2} \quad (36)$$

Similarly making use the identity Eq. (12) and setting a = 1 in Eq.(33), we possess the so-called bell type solitary wave solution of the nonlinear Fokker-Planck equation (33) as follows

$$u(x,t)=\frac{1}{2}\frac{-w-k+2Dk^2}{k}-4kD\,\mathrm{sech}^2 k(x-ct) \quad (37)$$

Making use of the identity Eq. (14) and setting a = -1 in Eq. (36), we find the singular travelling wave solution for the nonlinear Fokker-Planck equation (33) in the following form

$$u(x,t)=\frac{1}{2}\frac{-w-k+2Dk^2}{k}+4kD\,\mathrm{csch}^2 k(x-ct) \quad (38)$$

With the aid of the two previous equalities (16) and (18),then Eq.(37) and Eq.(38) admits

$$u(x,t)=\frac{1}{2}\frac{-w-k+2Dk^2}{k}-8kD\frac{1}{\cosh 2k(x-ct)+1} \quad (39)$$

$$u(x,t)=\frac{1}{2}\frac{-w-k+2Dk^2}{k}+8kD\frac{1}{\cosh 2k(x-ct)-1} \quad (40)$$

Making use of the identity Eq. (20),then Eq.(37) and Eq.(38) can be reduced to

$$u(x,t)=\frac{1}{2}\frac{-w-iK-2DK^2}{iK}-4iKD\sec^2 K(x-ct) \quad (41)$$

$$u(x,t)=\frac{1}{2}\frac{-w-iK-2DK^2}{iK}+4iKD\csc^2 K(x-ct) \quad (42)$$

which are two new triangle function-type periodic wave solutions of the nonlinear Fokker-Planck equation (33).

The generalized Burgers-Fisher equation: In this case we consider the generalized Burgers-Fisher equation [25] reads

$$u_t+pu^ru_x-u_{xx}-qu(1-u^r)=0 \quad (43)$$

where p and q are real constants and $r\neq 1$ is positive constants.

As stated before, inserting Eqs.(3-6) into Eq. (43) and collecting the coefficients of powers of y, give rise to a set of algebraic equations with respect to the unknown variables a and n. Solving the over-determined system of algebraic equation by virtue of Maple, we find that

$$a = a, u_0 = 1, n = -\frac{(4k^2 + q)}{k^2(3q - 4qk)}, w = 2k^2 + pk - q \qquad (44)$$

$$c = \frac{2k^2 + pk - q}{k} \qquad (45)$$

Substituting Eq.(44) into Eq.(6) and using Eq. (3), we get the general travelling wave solution to the generalized Buerger-Fisher equation (43) as follows

$$u(x,t) = 1 - \frac{4a(4k^2 + q)e^{2k(x-ct)}}{(3q - 4pk)(a + e^{2k(x-ct)})^2} \qquad (46)$$

By means of the identities Eq.(12),Eq. (14) and setting a = 1 and a = -1 into Eq.(46),we get the so-called bell-type solitary wave solutions of the generalized Burgers-Fisher equation (43) as

$$u(x,t) = 1 - \frac{(4k^2 + q)}{(3q - 4pk)}\operatorname{sech}^2 k(x - ct) \qquad (47)$$

$$u(x,t) = 1 + \frac{(4k^2 + q)}{(3q - 4pk)}\operatorname{csch}^2 k(x - ct) \qquad (48)$$

Knowing, Eq. (16) and Eq. (18), then Eq. (47) and Eq. (48), admits

$$u(x,t) = 1 - \frac{2(4k^2 + q)}{(3q - 4pk)}\frac{1}{\cosh 2k(x - ct) + 1} \qquad (49)$$

$$u(x,t) = 1 + \frac{2(4k^2 + q)}{(3q - 4pk)}\frac{1}{\cosh 2k(x - ct) - 1} \qquad (50)$$

By using Eq.(20), then Eq.(47) and Eq.(48) can be simplified as

$$u(x,t) = 1 - \frac{(-4K^2 + q)}{(3q - 4ipK)}\sec^2 K(x - ct) \qquad (51)$$

$$u(x,t) = 1 + \frac{(-4K^2 + q)}{(3q - 4ipK)}\csc^2 K(x - ct) \qquad (52)$$

which are two triangle function-type periodic wave solutions of the generalized Burgers-Fisher equation (43).

CONCLUSION

In summary, many types of explicit and exact travelling wave solutions to four nonlinear evolution equations arising in physics are obtained.

The validity of this method has been tested by applying it successfully to the generalized regularized long-wave (RLW) equation, the Camassa-Holm equation, nonlinear Fokker Planck equation and the generalized Burgers-Fisher equation.

As a result, these new exact travelling wave solutions include the solitary wave solutions, the singular travelling wave solutions and the triangle function-type periodic wave solutions, are successfully presented by making use of our unified trial function method.

Finally, it is worthwhile to mention that the proposed method is straightforward, concise and it is a promising and powerful new method for other nonlinear evolution equations in physics.

ACKNOWLEDGEMENTS

This paper have been supported by research support program, King Khalid University, Kingdom of Saudi Arabia, grand Number (KKU-COM-11-001).

REFERENCES

1. Hirota, R., 1971. Phys. Rev. Lett., 27: 1192.
2. Wadati, M., H. Sanuki and K. Konno, 1975. Prog. Theor. Phys., 53: 419.
3. He, J.H., 2006. Inter. J. of Modern Phys., B, 20 (10): 1141-1199.
4. El-Wakil, S.A. and M.A. Abdou, 2007. Chaos Solitons and Fractals, 33: 513-522.
5. Abdou, M.A., 2007. Chaos, Solitons and Fractals, 31: 195-104.
6. Abdou, M.A. and A.A. Soliman, 2005. Physica D, 211: 1.
7. El-Wakil, S.A. and M.A. Abdou, 2006. Elhanbaly A, Phys. Lett., A, 353: 40.
8. El-Wakil, S.A. and M.A. Abdou, 2007. Chaos, Solitons and Fractals, 31: 840-852.
9. Abdou, M.A. and A. Elhanbaly, 2006. Phys. Scripta, 73: 338.
10. El-Wakil, S.A. and M.A. Abdou, 2006. Phys. Lett. A, 358: 275-282.
11. Abdou, M.A., 2007. On the variational iteration method. Phys. Lett. A, 366: 61-68.
12. El-Wakil, S.A. and M.A. Abdou, 2007. New explicit and exact travelling wave solutions for two nonlinear evolution equations. J. of Nonlinear Dynamics, In Press.

13. Abdou, M.A., 2007. Generalized solitonary and periodic solutions for nonlinear partial differential equations by the Exp-function method. J. of Nonlinear Dynamics, In Press.
14. Dogan, K., 2004. Appl. Math. and Comput., 149: 833.
15. Peregine, D.H., 1996. J. Fluid Mech., 25: 321.
16. Peregine, D.H., 1967. J. Fluid Mech., 27: 815.
17. Benjamin, T.B., J.L. Bona and J.J. Mahony, 1972. Philos. Trans. Royal Soc. London, 227: 47.
18. Bona, J.L., W.G. Pritchard and L.R. Scott, 1983. Lectures in Appl. Math., pp: 235-267.
19. Fan, E., 2003. Chaos Solitons and Fractals, 16: 819.
20. Camassa, R. and D.D. Holm, 1993. Phys. Rev. Lett., 71: 1661.
21. Fuchssteiner, B. and A.S. Fokas, 1981. Physica D, 4: 47.
22. Donoso, J.M., J.J. Salgado and M. Soler, 1999. J. Phys. A: Math: Gen., 32: 3681.
23. Masoliver, J. and K. Wang, 1995. Phys. Review E, 51: 2987.
24. Klm, S., S. Hee Park and Su Ryu C, 1997. Phys. Lett. A, 230: 288.
25. Kaya, D. and S.M. El-Sayed, 2004. Appl. Math. and Comput., 152: 403.