

On the Numerical Solutions of Two Dimensional Maxwell's Equations

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Abstract: In this paper, two different methods are studied to solve numerically time dependent Maxwell's equations in two dimensions. The methods are, the Discontinuous Galerkin Method (DGM) and the Variational Iteration Method (VIM). Discontinuous Galerkin method is tested for different number of triangulation elements, different time intervals and different orders of basis functions. Comparisons between the solutions of Maxwell's equations using DGM and VIM are presented. It's found that VIM is invalid for large time, but DGM overcome this problem.

Key words: Discontinuous Galerkin method . time-domain maxwell's equations in two dimensions . variational iteration method

INTRODUCTION

Maxwell's equations are one of the important models in different fields; it describes electromagnetic phenomena such as includes micro, radio and radar, waves. It is well known that Maxwell's equations are hard to solve analytically, it can be solved analytically only for simple domains such as a sphere and an infinite circular cylinder [1]. Numerical methods for the Maxwell's equations are usually referred to as Computational Electromagnetic (CEM).

The modeling of systems are involving electromagnetic waves is now widely done through the solution of the time domain Maxwell equation on space grid. Such Systems were solved with many different methods, the first method for the numerical simulation of time dependent electromagnetic waves, the Finite Difference Time Domain Method (FDTD), was proposed in [2]. It is clear that FDTD, known due its simplicity and efficiency, but it is difficult to generalize to unstructured non-cartesian grids and suffer from the inaccurate representation of the solution on curved boundaries [2]. Moreover FDTD has accuracy limitation, for second order accuracy severely limits their ability to correctly represent wave motion over long distances unless the grid is prohibitively fine [3].

Many different types of methods have been proposed which are based on unstructured grids and can deal with complex geometries, like Finite Element Time Domain Methods (FETDMs) [4, 5]. There are two difficulties appear when using the standard Finite Element Method (FEM). First, the method generally

used on a globally conforming mesh, that mean, a mesh without hanging (connecting) nodes or mismatch of mesh points along internal boundaries. Second, how we can represent corner singularities [4, 6].

The Discontinuous Galerkin Methods (DGM) methods [7-11] are based on discontinuous finite element spaces. It is easily handle elements of various types and shapes, irregular non-conforming meshes and even locally varying polynomial degree. Moreover, continuity is weakly enforced across mesh interfaces by adding suitable bilinear forms (the so-called numerical fluxes) to the standard variational formulations [12]. Either on tetrahedral meshes using Lagrange polynomials [11, 13] or on hexahedral meshes using products of Lagrange polynomials [12].

On the other hand Variational Iteration Method (VIM), [14-20] is proposed by J. H. He [16, 17] as a modification of a general Lagrange multiplier method. This technique provides a sequence of functions which converges to the exact solution of the problem. It has been shown that this procedure is a powerful tool for solving various kinds of problems. This technique solves the problem without any need to discretization of the variables, therefore, it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time. Also, this technique provides the solution of the problem in a closed form while the mesh point techniques, such as the finite difference method provide the approximation at mesh points only.

This paper is organized as follows: In section 2, the model problem and the algorithm of solution are

introduced. In section 3, VIM is introduced to obtain the approximate solution of Maxwell's equations. The last section 4, gives a discussion of our results.

DGM FOR THE MODEL PROBLEM

Let us consider the two dimensional vacuum Maxwell's equations; which is known as Transverse Magnetic form (TM) [9]; defined in

$$\Omega = [-1, -1] \times [-1, -1], \quad t \in [0, T]$$

$$\begin{aligned} \mathbf{m}_r \frac{\partial \mathbf{H}^x}{\partial t} &= -\frac{\partial \mathbf{E}^z}{\partial y} \\ \mathbf{m}_r \frac{\partial \mathbf{H}^y}{\partial t} &= \frac{\partial \mathbf{E}^z}{\partial x} \\ \mathbf{e}_r \frac{\partial \mathbf{E}^z}{\partial t} &= \frac{\partial \mathbf{H}^y}{\partial x} - \frac{\partial \mathbf{H}^x}{\partial y} \end{aligned} \quad (2.1)$$

where, $\mathbf{H}^x(x, y, t)$, $\mathbf{H}^y(x, y, t)$, are the components of the magnetic field and $\mathbf{E}^z(x, y, t)$ is the electric field in z-direction. Furthermore, μ_r is called magnetic permeability and ϵ_r is called electric permittivity. For the boundary conditions, assume that the wall of the cavity is perfectly electrically conducting such that the tangential component of the electric field \mathbf{E}^z vanishes at the wall (i.e., $\mathbf{E}^z = 0$ at the walls), also the initial conditions of the problem are

$$\mathbf{H}^x(x, y, 0) = \mathbf{H}^y(x, y, 0) = 0$$

$$\mathbf{E}^z(x, y, 0) = \sin(m\pi x) \sin(n\pi y)$$

where m, n are arbitrary constants. For simplicity, let us write Maxwell's equation in the flux form [9]:

$$\mathbf{Q}(X) \frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{q}) = 0 \quad (2.2)$$

with

$$\mathbf{F}(\mathbf{q}) = [\mathbf{F}_1(\mathbf{q}), \mathbf{F}_2(\mathbf{q}), \mathbf{F}_3(\mathbf{q})]^T$$

$$\mathbf{Q}(x) = \begin{bmatrix} \mathbf{e}_r & 0 \\ 0 & \mathbf{m}_r \end{bmatrix}, \mathbf{q} = \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}$$

and

$$\mathbf{F}_i(\mathbf{q}) = \begin{bmatrix} -\mathbf{e}_i \times \mathbf{H} \\ -\mathbf{e}_i \times \mathbf{E} \end{bmatrix}$$

In the following we explain briefly the basic idea of DGM to solve (2.2), in the first part we present the space discretization and the second part we present the time discretization.

Space discretization: (a) Assume that the computational domain Ω is composed of K non-overlapping d-simplices or elements.

$$\Omega \approx \Omega_h = \bigcup_{k=1}^K D^k$$

where D^k is a straight sided triangle and the triangulation is assumed to be geometrically conforming; for example, $\partial\Omega$ is approximated by a piecewise linear polygon with each line segment being a face of triangle.

(b) Assume that the local solution of the vector of the field $\mathbf{q} = [\mathbf{E}, \mathbf{H}]^T$ and the local flux can be defined:

$$\begin{aligned} \mathbf{q}_h^k(X, t) &= \sum_{i=1}^{N_p} \mathbf{q}_h^k(X_i^k, t) l_i^k(X) \\ \mathbf{F}(\mathbf{q})_h^k(X, t) &= \sum_{i=1}^{N_p} \mathbf{F}(\mathbf{q})_h^k(X_i^k, t) l_i^k(X) \end{aligned} \quad (2.3)$$

where $l_i^k(X)$ is the multidimensional Lagrange polynomials defined by some grid points X_i and defined as,

$$l_i(X) \in P_p^2 = \text{span}\{x^i, y^j, i, j \geq 0; i+j \leq p\}$$

(c) To implement the numerical approximation, multiply equation (2.2) with test function then integrating by parts twice, this lead us to

$$\begin{aligned} \int_{D^k} \left[\frac{\partial \mathbf{q}_h^k}{\partial t} l_i^k(X) - \nabla \cdot \mathbf{F}_h^k \right] l_i^k(X) dX \\ = - \int_{\partial D^k} \hat{n} \cdot [\mathbf{F}_h^k - \mathbf{F}^*] l_i^k(X) dX \end{aligned} \quad (2.4)$$

which represent the strong form of DGM for problem (2.1).

(d) Assume that the local solution is expressed as

$$\begin{aligned} \mathbf{q}_h^k(X, t) &= \sum_{i=1}^{N_p} \mathbf{q}_h^k(X_i^k, t) l_i^k(X) \\ \sum_{i=1}^{N_p} \hat{\mathbf{q}}_h^k(X_i^k, t) \mathbf{y}_n(X), X \in D^k \end{aligned} \quad (2.5)$$

where $\{Y_n(X)\}_{n=1}^{N_p}$ is a genuine two dimensional polynomial basis of order N and N_p is the number of terms in the local expansion which related to the order of the polynomial with the relation,

$$N_p = \frac{(N+1)(N+2)}{2}.$$

(e) Then we have (2.3), (2.4) and (2.5) and after some rearrangement we get:

$$\begin{aligned} & \sum_{j=1}^{N_p} \frac{\partial q_h^k}{\partial t} \int_{D^k} l_i^k l_j^k dX + \sum_{j=1}^{N_p} F_h^k \cdot \int_{D^k} \nabla l_i^k l_j^k dX \\ & = \hat{n} \left(\sum_{j=1}^{N_p} (F_h^k - F^*) \right) \int_{D^k} l_i^k l_j^k dX \end{aligned} \quad (2.6)$$

which is the semi discrete form of Maxwell's equations, where the mass matrix,

$$M_{ij}^k = \int_{D^k} l_i^k l_j^k dX = J^k \int_1 l_i(r) l_j(r) dr \quad (2.7)$$

and it can be written as,

$$M^k = J^k (v v^T)^{-1}$$

Also the stiffness matrix,

Then the penalizing boundary term now defined as;

$$\hat{n} (F_N - \hat{F}) = \frac{1}{2} \left[\bar{Z}^{-1} (Z^+ \hat{n} \times [H_N] - a \hat{n} \times \hat{n} \times [E_N]) \right] \quad (2.10)$$

(f) Now the local semidiscrete scheme of Maxwell's equations getting after solving it using DGM and from (2.1) and (2.11) are given by:

$$\frac{d H_h^x}{dt} = -D_y E_h^z + \frac{1}{2} (JM)^{-1} \int_{D^k} \hat{n}_y [E_h^z] + a (\hat{n}_x [H_h] - [H_h^x] l(x)) dX \quad (2.11a)$$

$$\frac{d H_h^y}{dt} = D_x E_h^z + \frac{1}{2} (JM)^{-1} \int_{D^k} -\hat{n}_x [E_h^z] + a (\hat{n}_x [H_h] - [H_h^y] l(x)) dX \quad (2.11b)$$

$$\frac{d E_h^z}{dt} = D_x H_h^y - D_y H_h^x + \frac{1}{2} (JM)^{-1} \int_{D^k} (\hat{n}_y [H_h^x] - \hat{n}_x [H_h^y] + a E_h^z) l(x) dX \quad (2.11c)$$

Time discretization: To complete the semidiscrete formulation(2.11.a-2.11.c) use an explicit law storage Runge-Kutta method to integrate in time with p+1 stages (p is the order of the polynomial of the basis functions). The time step used in the computation is taken heuristically to be

$$\Delta t \leq CFL(p) \frac{\min_k h_k}{\max_k c_k} \quad \text{where} \quad c_k = \frac{1}{\sqrt{e_r m}}$$

$$S_{ij}^k = \int_{D^k} \nabla l_i^k l_j^k dX \quad (2.8)$$

where

$$S_{ij}^k = (S_{x,ij}^k, S_{y,ij}^k)$$

such that,

$$S_{x,ij}^k = \int_{D^k} \frac{\partial}{\partial x} l_i^k l_j^k dX$$

and

$$S_{y,ij}^k = \int_{D^k} \frac{\partial}{\partial y} l_i^k l_j^k dX$$

Finally the element of edge integration matrix

$$F_{ij}^k = \oint_{\partial D^k} l_i^k l_j^k dX = \hat{n} J^k \int_1 l_i(r) l_j(r) dr \quad (2.9)$$

and it will be calculated in the same way like the stiffness matrix, for more detailes for transformation and calculation see [7, 10, 12, 21].

Also to compute the suitable flux, we define,

$$Z^\pm = \frac{1}{Y^\pm} = \sqrt{\frac{m_r^\pm}{e_r^\pm}}, \bar{Z} = \frac{Z^- + Z^+}{2}, \bar{Y} = \frac{Y^- + Y^+}{2}$$

with boundaries,

$$\hat{n} \times E_N^- = -\hat{n} \times E_N^+$$

$$\hat{n} \times H_N^- = -\hat{n} \times H_N^+, Z^+ = Z^-$$

Table 1: The maximum error of the numerical solution at different values of p, k at different times t

T	$p = 4$	$p = 8$	$p = 10$	$p = 12$
At $k = 8$				
1	0.1077	3.0295e-004	7.7388e-006	1.3224e-007
5	0.0567	1.4393e-004	3.3340e-006	5.4983e-008
10	0.0724	1.9933e-004	4.8415e-006	8.1176e-008
20	0.0920	2.7861e-004	6.9832e-006	1.1918e-007
At $k = 46$				
1	0.0014	7.6734e-008	2.7618e-010	2.8005e-011
5	0.0023	1.5663e-007	6.2527e-010	3.1410e-011
10	0.0021	1.5826e-007	6.9626e-010	1.2363e-010
20	0.0018	1.2742e-007	1.5962e-009	4.6459e-010
At $k = 46$				
1	9.0438e-005	4.5008e-010	4.8844e-012	1.4435e-012
5	1.6846e-004	1.1097e-009	5.3945e-012	1.4656e-012
10	1.5260e-004	9.6749e-010	1.6166e-011	6.6469e-012
20	9.6523e-005	8.0295e-010	5.2121e-011	1.7011e-011

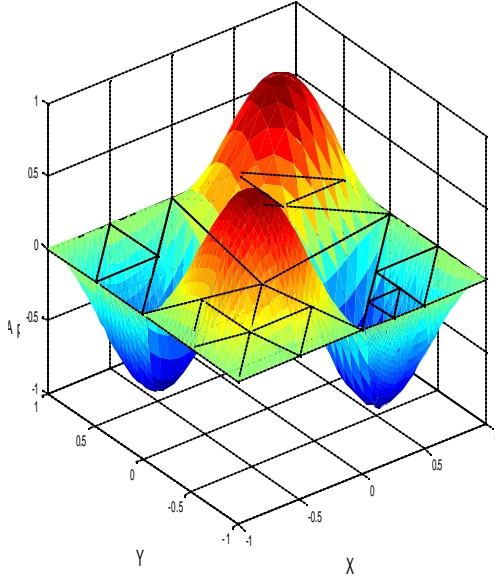


Fig. 1: Numerical solution for \tilde{E} using DGM at $t = 10$, $p = 12$ and $k = 8$

is the wave speed in the k -th element; h_k is the length (maximum diameter) of the element; and $CFL(p)$ condition typically takes values of $1/(1+2p)$.

The numerical results for solving Maxwell's equation using the DGM are shown in Table 1. This table presents the maximum error of the solution for different times t , number of elements K and the order of polynomial p .

NUMERICAL IMPLEMENTATION OF VIM

In this section, we apply VIM to obtain the approximate solution of (2.1) with the initial and conditions. According to VIM [16], we can construct the following iteration formula:

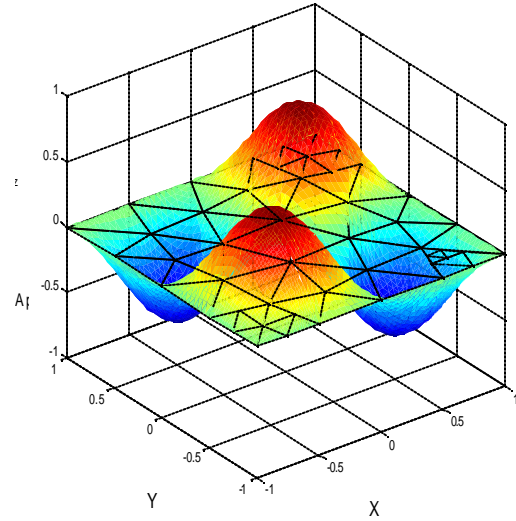


Fig. 2: Numerical solution for \tilde{E} using DGM at $t = 20$, $p = 12$ and $k = 46$

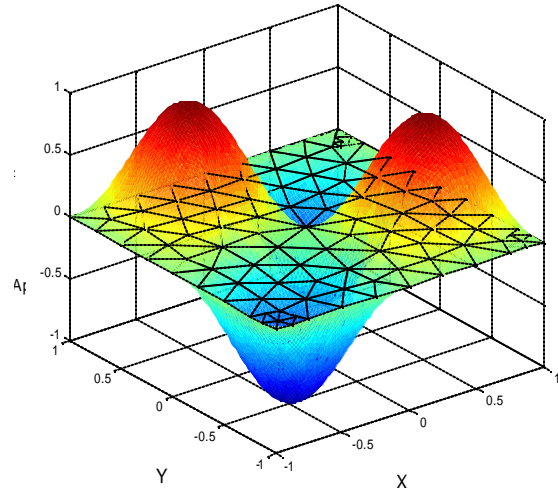


Fig. 3: Numerical solution for \tilde{E} using DGM at $t = 5$, $p = 12$ and $k = 146$

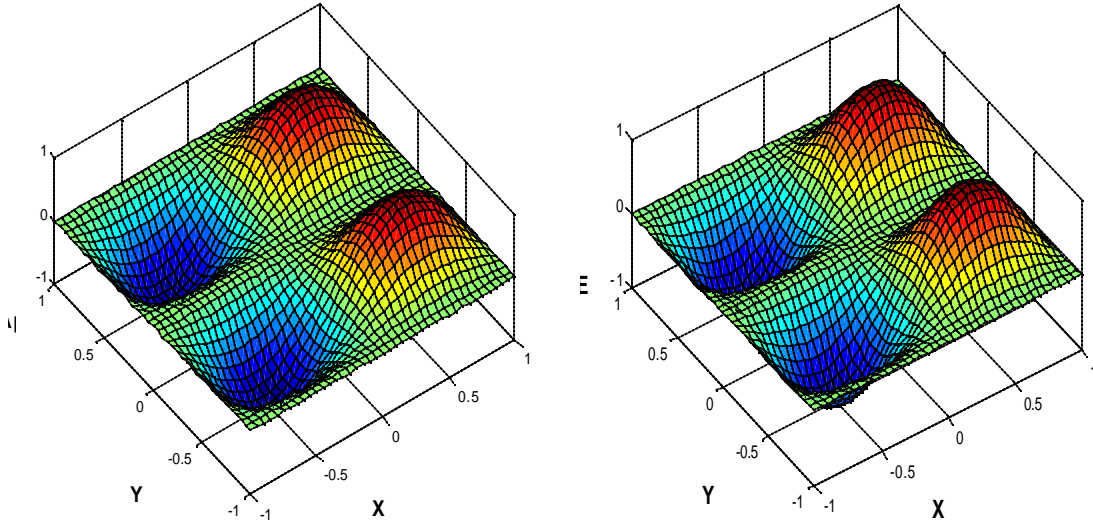


Fig. 4: The numerical solution using VIM at $t = 0.4$ after 60 iteration (left). Exact solution for E^z (right)

$$\begin{aligned} H_{n+1}^x(x, y, t) &= H_n^x(x, y, t) + \int_0^t I_1(t) \left[\frac{\partial H_n^x}{\partial t} + \frac{1}{\mathbf{m}_r} \frac{\partial E_n^z}{\partial y} \right] dt \\ H_{n+1}^y(x, y, t) &= H_n^y(x, y, t) + \int_0^t I_2(t) \left[\frac{\partial H_n^y}{\partial t} + \frac{1}{\mathbf{m}_r} \frac{\partial \tilde{E}_n^z}{\partial x} \right] dt \\ E_{n+1}^z(x, y, t) &= E_n^z(x, y, t) + \int_0^t I_3(t) \left[\frac{\partial E_n^y}{\partial t} - \frac{1}{\mathbf{e}_r} \frac{\partial \tilde{H}_n^y}{\partial x} + \frac{1}{\mathbf{e}_r} \frac{\partial \tilde{H}_n^x}{\partial y} \right] dt \end{aligned} \quad (3.1)$$

where λ_1 , λ_2 and λ_3 are general Lagrange multipliers, which can be identified optimally via variational theory. The second term on the right-hand side in (3.1) is called the correction and the subscript n denotes the n -th order approximation. Under a suitable restricted variational assumption (i.e. \tilde{H}_n^x , \tilde{H}_n^y and \tilde{E}_n^z is considered as a restricted variation), we can assume that the above correctional functional are stationary (i.e. $\delta \tilde{H}_n^x = 0$, $\delta \tilde{H}_n^y = 0$ and $\delta \tilde{E}_n^z = 0$), then the Lagrange multipliers can be identified [22]. The successive approximations H_n^x , H_n^y and E_{n+1}^z of the solution will be readily obtained upon using the obtained Lagrange multipliers and suitable selective functions H_0^x , H_0^y and E_0^z .

Calculating variation with respect to H_n^x , H_n^y and E_n^z we can obtain the following differential equations:

$$\dot{I}_1(t) = \dot{I}_2(t) = \dot{I}_3(t) = 0 \quad (3.2)$$

$$\begin{aligned} 1 + I_1(t) \Big|_{t=t} &= 1 + I_2(t) \Big|_{t=t} \\ &= 1 + I_2(t) \Big|_{t=t} = 0 \end{aligned}$$

The Lagrange multipliers, therefore, can be identified as:

$$I_1(t) = I_2(t) = I_3(t) = -1 \quad (3.3)$$

Substituting the identified multiplier (3.3) into (3.1) results the following iteration formula

$$\begin{aligned} H_{n+1}^x(x, y, t) &= H_n^x(x, y, t) - \int_0^t \left[\frac{\partial H_n^x}{\partial t} + \frac{1}{\mathbf{m}_r} \frac{\partial E_n^z}{\partial y} \right] dt \\ H_{n+1}^y(x, y, t) &= H_n^y(x, y, t) - \int_0^t \left[\frac{\partial H_n^y}{\partial t} + \frac{1}{\mathbf{m}_r} \frac{\partial E_n^z}{\partial x} \right] dt \end{aligned} \quad (3.4)$$

$$E_{n+1}^z(x, y, t) = E_n^z$$

$$- \int_0^t \left[\frac{\partial E_n^y}{\partial t} - \frac{1}{\mathbf{e}_r} \frac{\partial H_n^y}{\partial x} + \frac{1}{\mathbf{e}_r} \frac{\partial H_n^x}{\partial y} \right] dt$$

We can start with the given initial approximation using initial conditions and by the formula (3.4), some approximate solutions are listed below:

Table 2: The maximum error of the numerical solution at different values of times t

T	t = 0.0	t = 0.4	t = 0.8	t = 1.2	t = 1.6	t = 2.0
Max error	1.62114	0.0	1.55461	1.32534	1.39086	0.0
T	t = 2.4	t = 2.8	t = 3.2	t = 3.6	t = 4.0	t = 4.2
Max error	48.3274	0.0	1.57847×10 ⁶	297465.	2.7952×10 ⁶	4.5485×10 ⁶

$$E_0^z(x, y, t) = \sin(mpx) \sin(npy)$$

$$E_1^z(x, y, t) = \sin(mpx) \sin(npy)$$

$$E_2^z(x, y, t) = E_1^z(x, y, t) - \frac{m^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r} - \frac{n^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r}$$

$$E_3^z(x, y, t) = E_2^z(x, y, t) - \frac{m^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r} - \frac{n^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r}$$

$$E_4^z(x, y, t) = E_3^z(x, y, t) + \frac{(m^2 + n^2)^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r} - \frac{m^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r} - \frac{n^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r}$$

$$E_5^z(x, y, t) = E_4^z(x, y, t) + \frac{(m^2 + n^2)^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r} - \frac{m^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r} - \frac{n^2 p^2 t^2 \sin(mpx) \sin(npy)}{e_r m_r}$$

Hence the solution is may be obtained using:

$$E^z(x, y, t) = \lim_{n \rightarrow \infty} E_n^z(x, y, t) \quad (3.5)$$

found that VIM is invalid for large time and DGM overcome such this problem.

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The behavior of the approximate solution of the VIM and the exact solution is given in Fig. 4 at t = 0.4.

The numerical results for solving Maxwell's equation using the VIM are shown in Table 2. This table presents the maximum error of the solution for different times t.

CONCLUSION AND REMARKS

In this work the discontinuous Galerkin time domain method and VIM are used to study the numerical solution of Maxwell's equations in two-dimensions. The DGM is applied for different number of elements, different time and different order of polynomials. Here we the modes must be small enough for easy solution, if it being large give more oscillations, for small modes, give very good accuracy, also it must be integer not friction. Although, VIM is a very simple method and has many advantages such as it needs no discretization in space and time variables, no need to solve linear or nonlinear system of equations, but from the obtained numerical results in Table 2 we

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