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A Numeric-analytic Method for Approximating the Holling Tanner Model

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Abstract: This paper researches the accuracy of the Differential Transformation Method (DTM) for solving the Holling Tanner models which are described as two-dimensional system of ODES with quadratic and rational nonlinearities. Numerical results are compared to those obtained by the fourth-order Runge-Kutta method to illustrate the preciseness and effectiveness of the proposed method. The direct symbolic-numeric scheme is indicated to be efficient and accurate.

Key words:

INTRODUCTION

In this paper, we consider two different Holling-Tanner model: Model with a Wollkind *et al*. [1] and model with a Collings [2].

The May or Holling-Tanner model for predator-prey interaction is described by following differential equation system:

$$\frac{dN_1}{dT} = N_1\left[r\left(1-\frac{N_1}{K}\right)-\frac{cN_2}{m+N_1}\right]$$

$$\frac{dN_2}{dT} = sN_2\left(1-\frac{N_2}{nN_1}\right) \tag{1}$$

Here N_1 and N_2 denote prey and predator densities, respectively, in time T. It is assumed that in the absence of the predator, the prey population density grows according to a logistic curve with carrying capacity K and with an intrinsic growth rate (or biotic potential) r. The parameter s denotes the intrinsic growth rate of the predator. c is the maximal predator per capita consumption rate and m is the half capturing saturation constant. The predator growth equation is of logistic type with a modification of the conventional one. Here the available resources is not constant, but is equal to nN_1, which is proportional to prey abundance. The parameter n is the measure of the food quality that the prey provides for conversion into predator births. Several dynamical behaviours of Holling-Tanner model have been studied extensively in literature May [3], Tanner[4], Wollkind *et al*. [1], Murray [5], Hsu and Hwang [6], Collings [2, 7], Saez and Gonzalez-Olivares [8], Braza [9].

It is already mentioned that Haque and Li [10] have introduced a modified version of the above Holling-Tannermodel by replacing the Holling type-II prey-dependent functional response with a ratio-dependent one.

However, as a starting point of our study, we take their modified model described under the framework of the following set of ordinary differential equations:

$$\frac{dN_1}{dT} = N_1\left[r\left(1-\frac{N_1}{K}\right)-\frac{cN_2}{mN_2+N_1}\right]$$

$$\frac{dN_2}{dT} = sN_2\left(1-\frac{N_2}{nN_1}\right) \tag{2}$$

with

$$N_1(0)>0,\ N_2(0)>0,$$

$$\frac{dN_1}{dT} = 0 \text{ for } (N_1,N_2) = (0,0)$$

$$\frac{dN_2}{dT} = 0, N_1 = 0, r, K, m, s, c, n > 0$$

To reduce the number of parameters and to determine which combinations of parameters control the behaviour of the system, we nondimensionalize the system (2). We choose

$$x = \frac{N_1}{K}, y = \frac{mN_2}{K} \text{ and } t = rT$$

Then the system (2) takes the form (after some simplification)

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$$\frac{dx}{dt} = x(1-x) - \frac{\beta xy}{x+y}$$

$$\frac{dy}{dt} = \gamma y\left(1 - \frac{\delta y}{x}\right) \qquad (3)$$

with

$$x(0)>0,\ y(0)>0,$$

$$\frac{dx}{dt} = 0 \ \text{for}(x,y) = (0,0)$$

$$\frac{dy}{dt} = 0, x = 0, \qquad \beta = \frac{c}{mr}, \gamma = \frac{s}{r} \ \text{and}\ \delta = \frac{1}{mn}$$

where x and y are the dimensionless population variables; t is the dimensionless time variable; β, γ and δ are dimensionless parameters. Then, as in [11], we take $\beta = 1.8$, $\gamma = 0.2$ and $\delta = 1$

When dealing with nonlinear systems of ordinary differential equations, such as the Holling Tanner models, it is often difficult to obtain a closed form of the analytic solution. In the absence of such a solution, the accuracy of the DTM [12] is then tested against classical numerical methods, such as the Runge-Kutta method (RK4). RK4 has been widely and commonly used for simulating solutions for chaotic systems

The goal of this paper is to extend application to classical DTM and multi-step DTM for obtained approximant analytical solution of the aboved mentioned the Holling Tanner models.

DIFFERENTIAL TRANSFORMATION METHOD

Consider a general system of first-order ODES

$$\frac{dx_1}{dt} + h_1(t, x_1, x_2, \dots, x_m) = g_1(t)$$

$$\frac{dx_2}{dt} + h_2(t, x_1, x_2, \dots, x_m) = g_2(t)$$

$$\vdots$$

$$\frac{dx_m}{dt} + h_m(t, x_1, x_2, \dots, x_m) = g_m(t) \qquad (4)$$

subject to the initial conditions

$$x_1(t_0) = d_1, \quad x_2(t_0) = d_2, \cdots, x_m(t_0) = d_m \qquad (5)$$

To illustrate the Differential Transformation Method (DTM) for solving differential equations systems, the basic definitions of differential transformation are introduced as follows. Let x(t) be analytic in a domain D and let $t = t_0$ represent any point in D. The function x(t) is then represented by one power series whose center is located at t_0. The differential transformation of the *k* th derivative of a function x(t) is defined as follows:

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \forall t \in D \qquad (6)$$

In (6), x(t) is the original function and X(k) is the transformed function. As in [12-20] the differential inverse transformation of X(k) is defined as follows:

$$x(t) = \sum_{k=0}^{\infty} X(k)(t-t_i)^k, \forall t \in D \qquad (7)$$

From (6) and (7), we obtain

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \forall t \in D \qquad (8)$$

The fundamental theorems of the one-dimensional differential transform are:

Theorem 1: If $z(t) = x(t) \pm y(t)$, then $Z(k) = X(k) \pm Y(k)$.

Theorem 2: If $z(t) = cy(t)$, then $Z(k) = cY(k)$.

Theorem 3: If $z(t) = \frac{dy(t)}{dt}$, then $Z(k) = (k+1)Y(k+1)$.

Theorem 4: If $z(t) = \frac{d^n y(t)}{dt}$, then $Z(k) = \frac{(k+n)!}{k!}Y(k)$.

Theorem 5: If $z(t) = x(t)\, y(t)$, then

$$Z(k) = \sum_{k_1=0}^{k} X(k_1)Y(k-k_1).$$

Theorem 6: If $z(t) = t^n$, then $Z(k) = \delta(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$.

Theorem 7: If $z(t) = \frac{F(x(t),y(t))}{ax(t)+by(t)+c'}$, then

$$Z(k) = \frac{F\big(X(k), Y(k)\big) - a\sum_{k_1=1}^{k} X(k_1)Z(k-k1) - b\sum_{k_1=1}^{k} Y(k_1)Z(k-k1)}{(aX(0)+bY(0)+c)}$$

Theorem 8: If

$$z'(t) = \frac{F(x(t),y(t))}{ax(t)+by(t)+c'}$$

then $$Z(k+1) = \frac{F\big(X(k),Y(k)\big) - a\sum_{k_1=1}^{k}(k-k1+1)X(k_1)Z(k-k1+1) - b\sum_{k_1=1}^{k}(k-k1+1)Y(k_1)Z(k-k1+1)}{(aX(0)+bY(0)+c)(k+1)}$$

In real applications, the function x(t) is expressed by a finite series and (8) can be written as

$$x(t)=\sum_{k=0}^{N}X(k)(t-t_i)^k, \forall t\in D \tag{9}$$

Equation (9) implies that

$$\sum_{k=N+1}^{\infty}X(k)(t-t_i)^k$$

is negligibly small.

According to DTM, by taking differential transformed both sides of the systems of equations given Eq.(4) and (5) is transformed as follows:

$$\begin{aligned}(k+1)X_1(k+1)+H_1(k)&=G_1(k),\\(k+1)X_2(k+1)+H_2(k)&=G_2(k),\\&\vdots\\(k+1)X_m(k+1)+H_m(k)&=G_m(k).\end{aligned} \tag{10}$$

$$X_1(0)=d_1,\quad X_2(0)=d_2,\cdots,X_m(0)=d_m. \tag{11}$$

Therefore, according to DTM the n-term approximations for (4) can be expressed as

$$\varphi_{1,n}(t)=x_1(t)=\sum_{k=1}^{n}X_1(k)t^k, \tag{12}$$

$$\varphi_{2,n}(t)=x_2(t)=\sum_{k=1}^{n}X_2(k)t^k,$$

$$\vdots$$

$$\varphi_{m,n}(t)=x_m(t)=\sum_{k=1}^{n}X_m(k)t^k.$$

MULTI-STEP DIFFERENTIAL TRANSFORMATION METHOD

The approximate solutions (4) are generally, as will be shown in the numerical experiments of this paper, not valid for large t. A simple way of ensuring validity of the approximations for large t is to treat (10)-(11) as an algorithm for approximating the solutions of (4)-(5) in a sequence of intervals choosing the initial approximations as

$$\begin{aligned}x_{1,0}(t)&=x_1(t^*)=d_1^*,\\x_{2,0}(t)&=x_2(t^*)=d_2^*,\\&\vdots\\x_{m,0}(t)&=x_m(t^*)=d_m^*.\end{aligned} \tag{13}$$

In order to carry out the iterations in every subinterval $[0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$,…, $[t_{j-1}, t)$ of equal length h°, we would need to know the values of the following[15],

$$x_{1,0}^*(t)=x_1(t^*),\ x_{2,0}^*(t)=x_2(t^*),\ \ldots,\quad x_{m,0}^*(t)=x_m(t^*) \tag{14}$$

But, in general, we do not have these information at our clearance except at the initial point $t^*=t_0$. A simple way for obtaining the necessary values could be by means of the previous n-term approximations $\varphi_{1,n}$, $\varphi_{2,n}$,…, φ of the preceding subinterval, i.e.,

$$x_{1,0}^*\cong\varphi_{1,n}(t^*), x_{2,0}^*\cong\varphi_{2,n}(t^*),\ldots,x_{m,0}^*\cong\varphi_{m,n}(t^*). \tag{15}$$

RESULTS AND DISCUSSION

Taking the differential transformation of Eq. (3) with respect to time *t* gives

$$\begin{aligned}X(k+1)=\Bigg[&(1-\beta)\sum_{k_1=0}^{k}X(k_1)Y(k-k1)\\&-\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}X(k_1)X(k2-k1)X(k-k2)-\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}X(k_1)X(k2-k1)Y(k-k2)\\&+\sum_{k_1=0}^{k}X(k_1)X(k-k1)-\begin{matrix}\sum_{k_1=1}^{k}(k-k_1+1)X(k_1)X(k-k1+1)-\\\sum_{k_1=1}^{k}(k-k_1+1)X(k_1)Y(k-k1+1)\end{matrix}\Bigg]\Bigg/\Big(\big(Y(0)+X(0)\big)(k+1)\Big)\end{aligned} \tag{16}$$

$$Y(k+1)=\left[-\delta\gamma\sum_{k_1=0}^{k}Y(k_1)Y(k-k1)+\gamma\sum_{k_1=0}^{k}X(k_1)Y(k-k1)-\sum_{k_1=1}^{k}(k-k_1+1)X(k_1)Y(k-k1+1)\right]/(X(0)(k+1)) \tag{17}$$

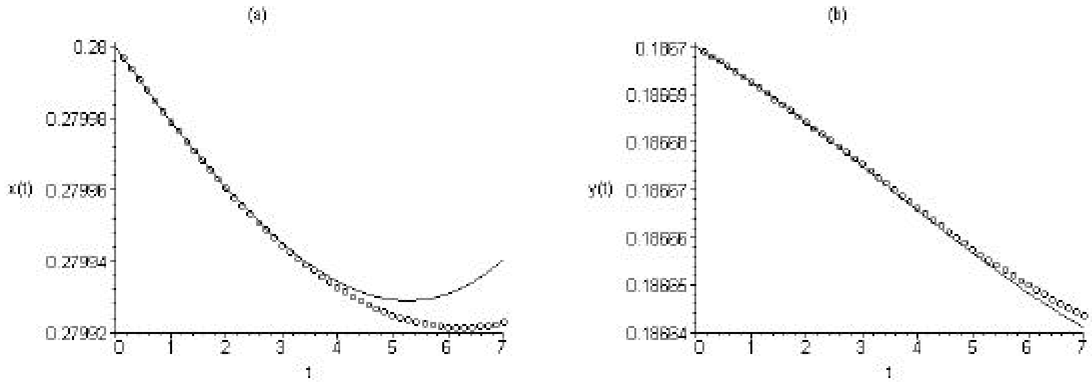

Fig. 1: Local changes of x, y for 3-term DTM(line) and RK4 with h = 0.001 (circle)

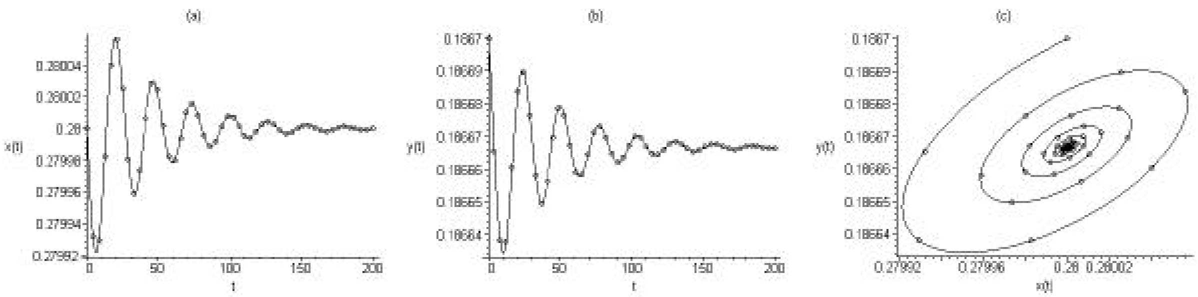

Fig. 2: Local changes and phase portrait of x, y for 3-term multi-step DTM with h = 0.1 (line) and RK4 with h = 0.001 (circle). Clearly it is a stable spiral converging to (X(0), Y(0)) = (0.28, 0.1867)

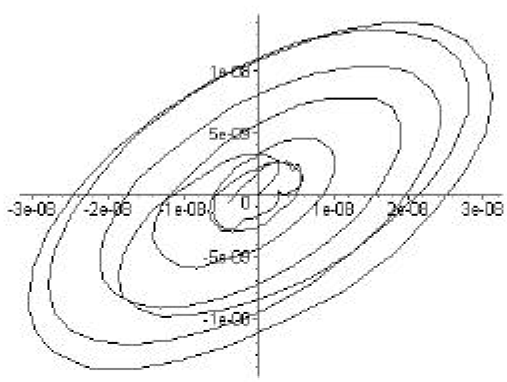

Fig. 3: Difference between 3-term DTM with Δt = 0.01 and RK4 with h = 0.001 with time span [0, 200]

where X(k) and Y(k) are the differential transformations of the corresponding functions x(t) and y(t), respectively and the initial conditions are given by X(0) = 0.28 and Y(0) = 0.1867.

CONCLUSION

In this paper, we apply the multi-step DTM, a reliable modification of the DTM, that improves the convergence of the series solution. The method provides immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations. The validity of the proposed method has been successful by applying it for the Holling-Tanner models. The method were used in a direct way without using linearization, perturbation or restrictive assumptions. It provides the solutions in terms of convergent series with easily computable components and the results have shown remarkable performance.

REFERENCES

1. Wollkind, D.J., J.B. Collings and J.A. Logan, 1988. Metastability in a temperature-dependent model system for predator-prey mite outbreak interactions on fruit trees. B. Math. Biol., 50: 379-409.
2. Collings, J.B., 1997. The effects of the functional response on the bifurcation behaviour of a mitepredator-prey interaction model. B. Math. Biol., 36: 149-168.
3. May, R.M., 1974. Stability and Complexity in Model Ecosystems. Princeton University Press, Princeton.

4. Tanner, J.T., 1975. The stability and intrinsic growth rates of prey and predator populations. Ecology, 56: 855-867.
5. Murray, J.D., 1993. Mathematical Biology, Springer-Verlag, New York.
6. Hsu, S.B. and T.W. Hwang, 1995. Global stability for a class of predator-prey systems. SIAM J. Appl. Math., 55: 763-783.
7. Collings, J.B., 1995. Bifurcation and stability analysis of a temperature-dependent mite predator-preyinteraction model incorporating a prey refuge. B. Math. Biol., 57: 63-76.
8. S´aez Gonz´alez-Olivares, E., 2003. Dynamics of a predator prey model. SIAM J. Appl. Math., 55: 1867-1878.
9. Braza, P.A., 2003. The bifurcation structure of the Hollinf-Tanner model for predator-prey interactionsusing two-timing. SIAM J. Appl. Math., 63: 889-904.
10. Haque, M. and B. Li, 2008. A ratio-dependent predator-prey model with logistic growth for the predatorpopulation. In: Proceedings of 10th International Conference on Computer Modeling and Simulation, University of Cambridge, Cambridge, UK, pp: 210-215.
11. Maiti, A. and S. Pathak, 2009. A Modified Holling-Tanner Model in Stochastic Environment. Nonlinear Analysis: Modelling and Control, 14: 51-71.
12. Zhou, J.K., 1986. Differential Transformation and its Applications for Electrical Circuits. Huazhong University Press, Wuhan, China, (In Chinese).
13. Ayaz, F., 2004. Solutions of the system of differential equations by differential transform method. Appl. Math. Comput., 147: 547-567.
14. Ayaz, F., 2004. Application of differential transform method to differential-algebraic equations. Appl. Math. Comput., 152: 649-657.
15. Odibat, Z.M., C. Bertelle, M.A. Aziz-Alaoui and G.H.E. Duchamp, 2010. A multi-step differential transform method and application to non-chaotic or chaotic systems. Computers and Mathematics with Applications, 259: 1462-1472.
16. Kanth, A. and K. Aruna, 2009. Two-dimensional differential transform method for solving linear and non-linear Schrödinger equations. Chaos Solitons Fractals, 41: 2277-2281.
17. Al-sawalha, M.M. and M.S.M. Noorani, 2009. A numeric-analytic method for approximating the chaotic Chen system. Chaos Solitons Fractals, 42: 1784-1791.
18. Arenas, A.J., G. González-Parra and B.M. Chen-Charpentier, 2009. Dynamical analysis of the transmission of seasonal diseases using the differential transformation method. Mathematical and Computer Modelling, 50: 765-776.
19. Yen-Liang Yeh, Cheng Chi Wang and Ming-Jyi Jang, 2007. Using finite difference and differential transformation method to analyze of large deflections of orthotropic rectangular plate problem. Appl. Math. Comp., 190: 1146-1156.
20. Abdel-Halim Hassan, I.H., 2008. Application to differential transformation method for solving systems of differential equations. Appl. Math. Modelling, 32: 2552-2559.