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Some Third-order Curvature Based Methods for Solving Nonlinear Equations

[1]Yong-Il Kim, [2]Changbum Chun and [3]Weonbae Kim

[1]School of Liberal Arts,
Korea University of Technology and Education, Cheonan, Chungnam 330-708, Republic of Korea
[2]Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea
[3]Department of Mathematics, Daejin University, Pocheon, Gyeonggi-do, 487-711, Republic of Korea

Abstract: In this paper we consider a curvature based scheme for the construction of iterative methods for the solution of nonlinear equations and a new class of methods from any iterative formulae of order at least two is presented. It is proven by analysis of convergence that each method of the class is at least third-order convergent. Our methods are tested on several numerical examples and their efficiency is demonstrated in comparison with Newton's method and the other third-order methods.

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Key words: Newton's method · iterative methods · nonlinear equations · order of convergence · circle of curvature

INTRODUCTION

This paper is concerned with iterative methods to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$ that uses no higher than the second derivative of f.

The best known iterative method for the calculation of α is Newton's method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

where x_0 is an initial approximation sufficiently close to α. This method is quadratically convergent [1].

There exists a modification of Newton's method with third-order convergence due to Potra and Ptak [2] defined by

$$x_{n+1} = x_n - \frac{f(x_n) + f\left(x_n - f(x_n)/f'(x_n)\right)}{f'(x_n)} \tag{2}$$

Some Newton-type methods with third-order convergence that do not require the computation of second derivatives have been developed [3-23]. To obtain some of those methods the Adomian decomposition method was applied in [3, 4], He's homotopy perturbation method in [5, 6] and Liao's homotopy analysis method in [7]. Other methods have been derived by considering different quadrature formulas for the computation of the integral arising from Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt \tag{3}$$

Weerakoon and Fernando [8] applied the rectangular and trapezoidal rules to the integral of (3) and obtained Newton's method and the cubically convergent method

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - f(x_n)/f'(x_n))} \tag{4}$$

while Frontini and Sormani [9] obtained the cubically convergenct method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/(2f'(x_n)))} \tag{5}$$

by considering the midpoint rule. In [10], Homeier derived the following cubically convergent iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/f'(x_n))}\right) \tag{6}$$

by considering Newton's theorem for the inverse function $x = (y)$ instead of $y = f(x)$. This scheme has

Corresponding Author: Dr. Changbum Chun, Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea

also been derived by Özban in [11] by using arithmetic mean of $f'(x_n)$ and $f'(x_n - f(x_n)/f'(x_n))$ instead of $f'(x_n)$ in Newton's method (1). On the other hand, Kou *et al*. in [12] considered Newton's theorem on a new interval of integration and arrived at the following cubically convergent iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n + f(x_n)/f'(x_n)) - f(x_n)}{f'(x_n)} \tag{7}$$

In this paper we also focus on developing the third-order modifications of Newton's method which improve any existing second-order formula in the order and are competitive with existing third-order methods in their efficiency. A detailed description of our construction from any given iteration formula of order two are provided and the resulted methods are presented in the following section. Finally, the comparison of our methods with other third-order methods is given.

ITERATIVE METHODS AND CONVERGENCE ANALYSIS

Let $y_n = f(x_n)$, $y'_n = f'(x_n)$ and $y''_n = f''(x_n)$, where x_n is an n-th iterate. To develop new methods we consider the circle of curvature, which has the same tangent line at the point (x_n, y_n) as the curve $y = f(x)$ and also a curve defined by the function $h(x) = g(x_n)(x-x_n)$ passing through the point $(x_n, 0)$ where g is a function to be determined later.

By an elementary calculation, the circle of curvature at (x_n, y_n) can be found

$$\left(x - x_n + \frac{y'_n[1+y'^2_n]}{y''_n}\right)^2 + \left(y - y_n - \frac{1+y'^2_n}{y''_n}\right)^2 = \frac{(1+y'^2_n)^3}{y''^2_n} \tag{8}$$

At the intersection point $(x_{n+1}, h(x_{n+1}))$ of the circle of curvature at (x_n, y_n) with the curve $y = h(x)$, we get

$$\left(x_{n+1} - x_n + \frac{y'_n[1+y'^2_n]}{y''_n}\right)^2 + \left(h(x_{n+1}) - y_n - \frac{1+y'^2_n}{y''_n}\right)^2 = \frac{(1+y'^2_n)^3}{y''^2_n} \tag{9}$$

Equation (8) can further be rewritten as follows

$$x_{n+1} = x_n - \frac{(h(x_{n+1}) - y_n)^2 - 2\dfrac{1+y'^2_n}{y''_n}(h(x_{n+1}) - y_n)}{x_{n+1} - x_n + 2\dfrac{y'_n(1+y'^2_n)}{y''_n}} \tag{10}$$

By replacing x_{n+1} on the right-hand side of (10) by the Newton iterate, we obtain the iterative method

$$x_{n+1} = x_n - \frac{(h(z_n) - y_n)^2 - 2\dfrac{1+y'^2_n}{y''_n}(h(z_n) - y_n)}{z_n - x_n + 2\dfrac{y'_n(1+y'^2_n)}{y''_n}} \tag{11}$$

where

$$z_n = x_n - \frac{y_n}{y'_n}$$

or

$$x_{n+1} = x_n - \frac{y'_n y''_n (h(z_n) - y_n)^2 - 2y'_n(1+y'^2_n)(h(z_n) - y_n)}{2y'^2_n(1+y'^2_n) - y_n y''_n} \tag{12}$$

Substituting

$$h(z_n) = g(x_n)(z_n - x_n) = -\frac{g(x_n)\, y_n}{y'_n}$$

into (12) gives

$$x_{n+1} = x_n - \frac{(g(x_n) + y'_n)[y''_n y_n (g(x_n) + y'_n) + 2y'_n(1+y'^2_n)]}{2y'^2_n(1+y'^2_n) - y_n y''_n}\frac{y_n}{y'_n} \tag{13}$$

If we compute the error equation for the iteration (13) by the help of Maple, we obtain

$$e_{n+1} = -\frac{g(\alpha)}{f'(\alpha)}e_n - \frac{1}{2}\frac{\begin{array}{c} f''(\alpha)g(\alpha)[f'(\alpha)g(\alpha) - f'^2(\alpha) - 2] \\ +2f'(\alpha)g'(\alpha)[1+f'^2(\alpha)] \end{array}}{f'^2(\alpha)[1+f'^2(\alpha)]}e_n^2 + C(g(\alpha), f'(\alpha), f''(\alpha), f'''(\alpha))e_n^3 + O(e_n^4) \tag{14}$$

where $e_n = x_n - \alpha$. Thus, for any real valued function g satisfying the conditions

$$g(\alpha) = g'(\alpha) = 0 \tag{15}$$

the iteration (13) yields a third-order modification of Newton's method. Many choices of g are possible to obtain iterative methods. If we take $g \equiv 0$, then (13) reduces to the method presented in [18]. One may take $g(x) = f^m(x)$, $m \geq 2$, giving

$$x_{n+1} = x_n - \frac{(f^m(x_n) + y'_n)\left[\begin{array}{r} y''_n y_n (f^m(x_n) + y'_n) \\ +2y'_n(1+y'^2_n) \end{array}\right]}{2y'^2_n[1+y'^2_n] - y_n y''_n}\frac{y_n}{y'_n} \tag{16}$$

Of particular interest among those choices is to take $g(x) = \psi(x) \equiv (x - \phi(x))^m$, $m \geq 2$, where ϕ is any given iteration function of order at least two. This gives rise to a class of third-order methods

$$x_{n+1} = x_n - \frac{(\psi(x_n) + y'_n)[y''_n y_n (\psi(x_n) + y'_n) + 2y'_n (1 + y'^2_n)]}{2y'^2_n[1 + y'^2_n] - y_n y''_n} \frac{y_n}{y'_n} \tag{17}$$

If we consider an iteration function ϕ of order two that requires the computation of the functions y_n, y'_n and y''_n, then new third-order method of practical utility results in.

We observe that the method (13) and (17) require evaluation of the second derivative. To derive its second-derivative-free variant, which is important from the practical point of view, we consider the approximation

$$y''_n \approx \frac{f'(z_n) - y'_n}{z_n - x_n} = -\frac{y'_n}{y_n}(f'(z_n) - y'_n) \tag{18}$$

where, $z_n = x_n - \frac{y_n}{y'_n}$ to obtain second-derivative-free iteration formula

$$x_{n+1} = x_n - \frac{(\psi(x_n) + y'_n)[2(1 + y'^2_n) - (f'(z_n) - y'_n)(\psi(x_n) + y'_n)]}{f'(z_n) + y'_n(1 + 2y'^2_n)} \frac{y_n}{y'_n} \tag{19}$$

We can prove that formula (19) also has the order of convergence three. If we consider an iteration function ϕ of order two requiring the evaluation of the functions y_n, y'_n and $f'(z_n)$, (19) yields a new third-order method, which is more optimal than the original one in the efficiency.

If we consider the Newton iteration function defined by $\phi(x) = x - \frac{f(x)}{f'(x)}$ in (19), then we obtain the new third-order method

$$x_{n+1} = x_n - \frac{\left(\frac{f^m(x_n)}{f^m(x_n)} + f'(x_n)\right)\left[2(1 + f'^2(x_n)) - (f'(z_n) - f'(x_n))\left(\frac{f^m(x_n)}{f'^m(x_n)} + f'(x_n)\right)\right]}{f'(z_n) + f'(x_n)(1 + 2f'^2(x_n))} \cdot \frac{f(x_n)}{f'(x_n)} \tag{20}$$

where, $z_n = x_n - \frac{f(x_n)}{f'(x_n)}$. If we consider the second-order iteration function [16] defined by $\phi(x) = x - \frac{f(x)}{f(x) + f'(x)}$ in (19), then we obtain the new third-order method

$$x_{n+1} = x_n - \frac{(y_n + y'_n y'_n + y'^2_n)[2(1 + y'^2_n)(y_n + y'_n) - (f'(z_n) - y'_n)(y_n + y_n y'_n + y'^2_n)]}{(y_n^2 + y'^2_n)[f'(z_n) + y'_n(1 + 2y'^2_n)]} \frac{y_n}{y'_n} \tag{21}$$

where, $y_n = f(x_n), y'_n = f'(x_n)$. If we consider the second-order iteration function [17] defined by $\phi(x) = x - \frac{f(x)f'(x)}{f^2(x) + f'^2(x)}$ in (19), then we obtain the new third-order method

$$x_{n+1} = x_n - \frac{(y_n y'_n + y_n^2 y'_n + y'^3_n)[2(1 + y'^2_n)(y_n^2 + y'^2_n) - (f'(z_n) - y'_n)(y_n y'_n + y_n^2 y'_n + y'^3_n)]}{(y_n^2 + y'^2_n)[f'(z_n) + y'_n(1 + 2y'^2_n)]} \frac{y_n}{y'_n} \tag{22}$$

where, $y_n = f(x_n), y'_n = f'(x_n)$. Thus, we have proved the following theorem:

Theorem 2.1: Let $\alpha \in I$ be a simple zero of sufficiently differentiable function f: $I \mapsto R$ for an open interval I. If x_0 is sufficiently close to α and g satisfies conditions g $(\alpha) = g'(\alpha) = 0$, then the order of convergence of the methods defined by (13) is three.

NUMERICAL EXAMPLES

We present some numerical test results for various cubically convergent iterative methods in Table 1-7.

The following methods were compared: the Newton method (NM), the method of Weerakoon and Fernando (4) (WF), the midpoint method (5) (MP), Homeier's method (6) (HM), the method of Kou *et al*. (7) (KM) and our new curvature method (20) with m=3 (KCM), which was arbitrarily chosen from the methods presented in this contribution. All computations were done using MAPLE using 64 digit floating point arithmetic (Digits:=64). We accept an approximate solution rather than the exact root, depending on the precision (ε) of the computer. We use the following stopping criteria for computer programs:

(i) $|x_{n+1} - x_n| < \varepsilon$

(ii) $|f(x_{n+1})| < \varepsilon$

and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. For numerical illustrations in this section we used the fixed stopping criterion $\varepsilon = 10^{-15}$. We used the following test functions and display the computed approximate zeros x_*:

$$f_1(x) = x^3 + 4x^2 - 10,$$
$$x_* = 1.365230013414096845760806829O$$

$$f_2(x) = \sin^2(x) - x^2 + 1,$$
$$x_* = 1.404491648215341226035086817 8$$

$$f_3(x) = x^2 - e^x - 3x + 2,$$
$$x_* = 0.257530285439860760455367304 94$$

$$f_4(x) = \cos x - x,$$
$$x_* = 0.739085133215160641655312087 67$$

$$f_5(x) = (x-1)^3 - 1, \quad x_* = 2$$

$$f_6(x) = \sin x - x/2,$$
$$x_* = 1.895494267033980947144035738 1$$

$$f_7(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$$
$$x_* = -1.207647827130918927009416758 4.$$

As convergence criterion, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-15}. Also displayed are the number of iterations to approximate the zero (IT), the number of functional evaluations (NFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative and the value $f(x_*)$. The numerical results presented in the tables

Table 1: $f_1(x) = x^3+4x^2-10$, $x_0 = 3.0$

	IT	NFE	$f(x_*)$	δ
NM	7	14	4.60e-39	2.38e-20
WF	5	15	0	2.96e-24
MP	5	15	0	2.25e-26
HM	5	15	0	1.80e-42
KM	5	15	0	2.48e-22
KCM	5	15	8.49e-58	4.65e-20

Table 2: $f_2(x) = \sin^2(x)-x^2+1$, $x_0 = 3.5$

	IT	NFE	$f(x_*)$	δ
NM	7	14	-3.03e-43	3.95e-22
WF	5	15	-2.0e-63	2.12e-30
MP	5	15	-4.56e-61	6.76e-21
HM	5	15	-2.0e-63	9.30e-33
KM	5	15	1.18e-45	7.67e-16
KCM	7	21	-1.41e-59	1.68e-20

Table 3: $f_3(x) = x^2-e^x-3x+2$, $x_0 = -1.0$

	IT	NFE	$f(x_*)$	δ
NM	6	12	1.10e-52	1.76e-26
WF	4	12	3.72e-54	2.98e-18
MP	4	12	0	3.26e-24
HM	4	12	1.0e-63	3.79e-22
KM	4	12	-1.77e-54	1.69e-18
KCM	5	15	1.0e-63	1.63e-32

Table 4: $f_4(x) = \cos x-x$, $x_0 = 1.2$

	IT	NFE	$f(x_*)$	δ
NM	5	10	-1.90e-35	7.16e-18
WF	4	12	0	1.97e-34
MP	4	12	0	2.72e-27
HM	4	12	0	4.0e-29
KM	4	12	-6.07e-57	2.50e-19
KCM	4	12	-3.37e-62	7.31e-21

Table 5: $f_5(x) = (x-1)^3-1$, $x_0 = 2.5$

	IT	NFE	$f(x_*)$	δ
NM	7	14	5.03e-56	1.29e-28
WF	5	15	0	4.57e-34
MP	5	15	0	4.46e-37
HM	4	12	5.98e-54	2.29e-18
KM	5	15	0	7.99e-33
KCM	5	15	0	2.02e-28

Table 6: $f_6(x) = \sin x\ x/2$, $x_0 = 1.3$

	IT	NFE	$f(x_*)$	δ
NM	10	20	-1.25e-44	1.21e-22
WF	5	15	2.0e-64	2.47e-23
MP	5	15	7.98e-50	4.66e-17
HM	divergent			
KM	18	54	0	2.11e-31
KCM	divergent			

Table 7: $f_7(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$, , x_0 = -2.0

	IT	NFE	$f(x_*)$	δ
NM	9	18	-2.27e-40	2.73e-21
WF	7	21	-4.0e-63	3.11e-44
MP	6	18	-4.0e-63	2.12e-23
HM	6	18	-4.0e-63	2.57e-32
KM	6	18	-4.0e-63	8.87e-34
KCM	7	21	-4.0e-63	4.45e-30

shown below demonstrate the efficiency of one of the methods presented in this work. Notice that the number of iterations (IT) is given along with the value of the function at the last iteration $f(x_n)$. From the numerical tests we performed we conclude that the newly presented methods can be competitive with Newton's method, which is representative of classical methods and other third-order methods in the literature.

CONCLUSION

In this paper, we presented general third-order methods for solving nonlinear equations based on the circle of curvature and an auxiliary curve. These methods have the same efficiency as the other third-order methods in the literature. We conclude from numerical examples that the proposed methods have at least equal performance as compared with the other methods of the same order.

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