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Long Time Dynamics of Forced Oscillations of the Korteweg-de Vries Equation Using Homotopy Perturbation Method

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Abstract: In this paper we study the long time dynamics of solutions of Initial and Boundary Value Problem (IBVP) of the Korteweg-de Vries (hence-forth KdV) equation posed on a bounded domain:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & u(x,0) = f(x), \quad 0 < x < 1, \; t > 0, \\ u(0,t) = h(t), \quad u(1,t) = 0, & u_x(1,t) = 0, \quad t > 0. \end{cases} \tag{*}$$

It is shown, using Homotopy Perturbation Method (HPM), that if the boundary forcing h is periodic of period τ, then the solution u (forced oscillations) of the IBVP (*) at each spatial point becomes eventually time-periodic of period τ. In order to exhibit eventual periodicity we approximate the solution of the IBVP analytically using Homotopy Perturbation Method. To confirm our theoretical results we present some numerical experiments using Mathematica.

Key words: Korteweg-de vries equation . HPM . forced oscillations . eventual periodicity . nonlinear dispersive wave equation

INTRODUCTION

Various physical phenomena in physics and engineering may be described by nonlinear ordinary or partial differential equations. Most of them are difficult to solve either theoretically or numerically. Recently there has been much attention devoted to the search of better and computationally efficient methods for determining a solution, approximate or exact [2, 10, 13], analytical or numerical [2, 5], to nonlinear models. There are many methods available to solve nonlinear ordinary or partial differential equations, for example, grid-based finite difference and finite element methods. Recently Homotopy Perturbation Methods [1, 3, 6, 7,9] have gained a lot of attention in the scientific community. This method presents an accurate and stable analytical approximate solution for integral equations or PDEs with a variety of boundary conditions. In traditional FDM and FEM, mesh generation for the problem domain is a pre-requisite for the numerical solutions. In some cases it becomes more expensive than solving the problem itself.

In many physical systems, which are nonlinear, dispersive and dissipative the asymptotic analysis of the governing differential equations in far-field and long-wave approximations leads to the Korteweg-de Vries (KdV) equation. KdV equation is initially introduced to describe the lossless propagation of shallow water waves. It represents the long time evolution of wave phe-nomena, in which the effect of the nonlinear term is counterbalanced by the dispersion term.

In this paper, we study an Initial Boundary Value Problem (IBVP) for the Korteweg-de Vries (KdV) equation of the type

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & u(x,0) = f(x), \quad 0 < x < 1, \; t > 0, \\ u(0,t) = h(t), \quad u(1,t) = 0, & u_x(1,t) = 0, \quad t > 0. \end{cases} \tag{1.1}$$

posed on a finite domain (0, 1). Guided by the outcome of laboratory experiments, interest is given to the long time effect of the boundary forcing h and large time behavior of solutions of IBVP (1.1).

In the experiments of Bona, Pritchard and Scott [4], a channel partly filled with water was mounted with a flap-type wave maker at one end. Each experiment commenced with water in the channel at rest. The wave-maker was activated and underwent periodic oscillations. The throw of the wave-maker and its frequency of oscillation were such that the surface waves brought into existence were of small amplitude

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and long wavelength, so it could be modeled by a Korteweg-de Vries (KdV) type equation posed in a quarter plane:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} - a u_{xx} - gu = 0, \; x > 0, t > 0 \\ u(x,0) = 0, \quad u(0,t) = h(t), \; x \geq 0, \; t > 0 \end{cases} \qquad (1.2)$$

where α and γ are nonnegative constants that are proportional to the strength of the damping effect. The wave-maker is modeled by the boundary value function $h = h(t)$ which is assumed to be a periodic function of period τ.

It was observed in the experiments that at each fixed station down the channel, for example at a spatial point x_0, the wave motion $u(x_0, t)$ rapidly became periodic of the same period as that of the boundary forcing function $h(t)$. This observation led to the following conjecture for solutions of IBVP (1.2).

Conjecture If the boundary forcing h is a periodic function of period τ, then the solution u of IBVP (1.2) eventually becomes time-periodic of period τ, i.e.,

$$\lim_{t\to\infty} \left\| u(\cdot, t+t) - u(\cdot, t) \right\|_{L^2(R^+)} = 0$$

Recently, the IBVP for KdV type equations on finite domain have received much attention. The KdV equation is a model equation for water waves in a channel using long-wave approximation and these equations are usually considered on unbounded domains. But the fluid regions are always finite in most of the physical applications and fluid dynamical experiments as well as numerical computations. So it is more realistic to consider the KdV equation on a bounded domain.

In the most of the previous related works, it was assumed that one or both of the damping coefficients α and γ in (1.2) are greater than zero. More recently, Zhang and Usman [11, 12], along with other results, proved that without any explicit damping term (both $\alpha = 0$, $\gamma = 0$), solution u of the IBVP (1.1) posed on a finite domain (0, 1), is asymptotically time-periodic if the boundary forcing h is periodic with small amplitude and the initial data $f(x)$ is small in certain space. In (1.1) the wave-maker is putting energy in a finite channel from the left boundary ($x = 0$) while the right end ($x = 1$) of the channel is free. In fact, if we set $h(t) = 0$ along with both $\alpha = \gamma = 0$ then by multiplying the equation by u, integration by parts and the boundary conditions yield

$$\frac{d}{dt}\int_0^1 u^2(x,t)\,dx = -u_x^2(0,t)$$

for any $t \geq 0$. This shows that there is a weak dissipation from the left end of the boundary, although there is no explicit dissipation in the system.

In this paper, we study the IBVP (1.1) for the KdV equation on a bounded domain. We obtain an analytical solution of the problem using the Homotopy Perturbation method [7, 8] and study the eventually periodicity of the solution. We show that the solution obtained is time periodic with the same period as that of the boundary function h. We do not require the amplitude of the boundary function to be small which is crucial in the theoretical results [11].

The paper is organized as follows: Section 2 is devoted to analytical approximate solution of IBVP of the KdV equation on a bounded domain using the Homotopy Perturbation method. Finally the numerical experiments using Mathematica are presented in section 3.

ANALYTICAL SOLUTION AND EVENTUAL PERIODICITY

In the literature there are many mathematical methods available to solve nonlinear ordinary and partial differential equations. However, most of them require tedious analysis or large memory computers to handle numerical techniques. Recently Ji-Huan He [6-8] suggested some new approximate analytical methods overcoming the shortcomings of many available methods.

In order to obtain an analytical approximate solution of the IBVP (1.1) for KdV equation, consider a one-parameter family of IBVPs

$$\begin{cases} u_{xxx} + p\left[u_t + u_x + uu_x\right] = 0, & x \in (0,1) \\ u(x,0) = f(x), \quad x \in (0,1) \\ u(0,t) = h(t), \quad u(1,t) = 0, \quad u_x(1,t) = 0 \end{cases} \qquad (2.1)$$

where the parameter $p \in [0,1]$.
If $p = 0$, we have a simple linear problem

$$\begin{cases} u_{xxx} = 0, & x \in (0,1) \\ u(x,0) = f(x), \quad x \in (0,1) \\ u(0,t) = h(t), \quad u(1,t) = 0, \quad u_x(1,t) = 0 \end{cases} \qquad (2.1)$$

whose solution is

$$u_0(x,t) = h(t)(1-x)^2 \qquad (2.3)$$

If $p = 1$, we have the original problem (1.1). The one parameter family of problems (2.1) is also known as homotopy. Let the solution u(x, t) of the system (2.1) be expressed as an infinite series, that is,

$$u(x,t)=\sum_{i=0}^{\infty} u_i\ (x,t)p^i \tag{2.4}$$

Then

$$\sum_{i=0}^{\infty} u_i\ (x,t)$$

is a series solution of the IBVP (1.1), where the components $u_i(x,t)$ are such that

$$\sum_{i=0}^{\infty} u_i\ (x,0)=f(x)$$

Substituting (2.4) in (2.1), we have

$$\begin{cases} \sum_{i=0}^{\infty} u_{i,xxx}\, p^i + p\left[\sum_{i=0}^{\infty}(u_{i,t}+u_{i,x})p^i + \sum_{i=0}^{\infty} u_i p^i . \sum_{i=0}^{\infty} u_{\ i,x} p^i\right]=0 \\ \sum_{i=0}^{\infty} u_i\ (x,0)p^i=\phi(x), \qquad x\in(0,1) \\ \sum_{i=0}^{\infty} u_i\ (0,t)p^i=h(t), \sum_{i=0}^{\infty} u_i(1,t)p^i=0, \sum_{i=0}^{\infty} u_{i,x}\ (1,t)p^i=0 \end{cases} \tag{2.5}$$

where $u_{i,z}$ denotes the derivative of u_i with respect to z. Equating coefficients of p, p^2,…, we obtain

$$\begin{cases} u_{1,xxx}+(u_{0,t}+u_{0,x})+u_0u_{0,x}=0,\ x\in(0,1) \\ u_1(0,t)=0, \quad u_1(1,t)=0, \quad u_{1,x}(1,t)=0 \end{cases} \tag{2.6}$$

$$\begin{cases} u_{2,xxx}+(u_{1,t}+u_{1,x})+u_0u_{1,x}+u_{0,x}u_1=0,\ x\in(0,1) \\ u_2(0,t)=0, \quad u_2(1,t)=0, \quad u_{2,x}(1,t)=0 \end{cases} \tag{2.7}$$

and so on. Solving (2.6) and (2.7), we obtain

$$\begin{aligned} u_1(x,t)&=\frac{h(t)}{12}\left[(1-x)^2-(1-x)^4\right] \\ &\quad+\left(\frac{h^3(t)}{30}-\frac{h'(t)}{60}\right)\left[(1-x)^2-(1-x)^5\right] \end{aligned} \tag{2.8}$$

$$\begin{aligned} u_2(x,t)&=\frac{1}{24}\left(\frac{h(t)}{6}+\frac{h'^{3}(t)}{15}-\frac{h'(t)}{30}\right)\left[(1-x)^2-(1-x)^4\right] \\ &+\frac{1}{24}\left(\frac{h''(t)}{60}-\frac{h^2(t)h'(t)}{10}-\frac{h'(t)}{12}\right)\left[(1-x)^2-(1-x)^5\right] \\ &+\frac{1}{120}\left(\frac{h^2(t)}{3}-\frac{h(t)}{3}-\frac{h(t)h'(t)}{15}+\frac{2h'^{4}(t)}{15}\right)\left[(1-x)^2-(1-x)^6\right] \\ &+\frac{1}{210}\left(\frac{h(t)}{6}-\frac{h^3(t)}{6}\right)\left[(1-x)^2-(1-x)^7\right] \\ &+\frac{1}{336}\left(-\frac{h^2(t)}{2}-\frac{h'''(t)}{60}+\frac{h^2(t)h'(t)}{10}\right)\left[(1-x)^2-(1-x)^8\right] \\ &+\frac{1}{504}\left(-\frac{7h'^{4}(t)}{30}+\frac{7h(t)h'(t)}{60}\right)\left[(1-x)^2-(1-x)^9\right] \end{aligned} \tag{2.9}$$

Hence, three terms approximate solution of the IBVP (1.1) for the KdV equation is

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)$$

where u_0, u_1 and u_2 are given by (2.3), (2.8) and (2.9) respectively.

Now we present the plots of solutions at different spatial points to show the time periodicity.

RESULTS AND DISCUSSION

In this section we present the numerical experiments investigating the eventual periodicity of the solutions for (1.1) with periodic forcing $h(t)$. Figure 1 is the 3D and contour plots of $u(x, t)$ with $h(t) = 0.1$ sin(2pt). Figures 2 are plots of $u(x, t)$ for $x = 0.5$ and 0.7 showing eventual periodicity. In Fig. 3 the left graph shows the time versus amplitude plots of u_0, u_1, u_2, u_3 and h. Fig. 3, right plot shows $u(x, t)$ at $x = 0.5$ including the another term u_3 of the series solution exhibiting the eventual periodicity.

In our second set of numerical experiment Fig. 4 displays the 3D and contour plots of $u(x, t)$ with forcing $h(t) = 10$ sin(2pt). Figure 5 shows eventual periodicity of $u(x, t)$ at $x = 0.5$ and $x = 0.7$, when the boundary forcing has large amplitude of 10. In Fig. 6 the left graph shows the time versus amplitude plots of u_0, u_1, u_2, u_3 and h. We have computed the next order term u_3 in the solution, using Mathematica and due to length of the expression we have not presented its analytical expression. This term is small in order and does not make significant difference. In Fig. 6 right plot shows $u(x, t)$ at $x = 0.5$ including the term u_3 of the series solution exhibiting the eventual periodicity. All of the numerical simulations are presented here are for the time interval such that the wave-front generated by the boundary forcing has not reached at the right end $x = 1$.

Remarks: In this paper we have studied analytical solutions for the initial-boundary-value problem of the KdV equation posed on a finite interval (0, 1):

$$\begin{cases} u_t+u_x+uu_x+u_{xxx}=0, \quad 0<x<1, \quad t>0 \\ u(x,0)=\phi(x) \\ u(0,t)=h(t),\ u(1,t)=0,\ u_x(1,t)=0,\ 0<x<1,\ t>0 \end{cases} \tag{3.1}$$

We have applied the Homotopy Perturbation (HP) method to obtain the solution of an IBVP of the KdV equation in terms of a convergent series with easily computable components. It is shown that if the boundary forcing is periodic, then any solution of (3.1) is asymptotically time periodic of the same period as

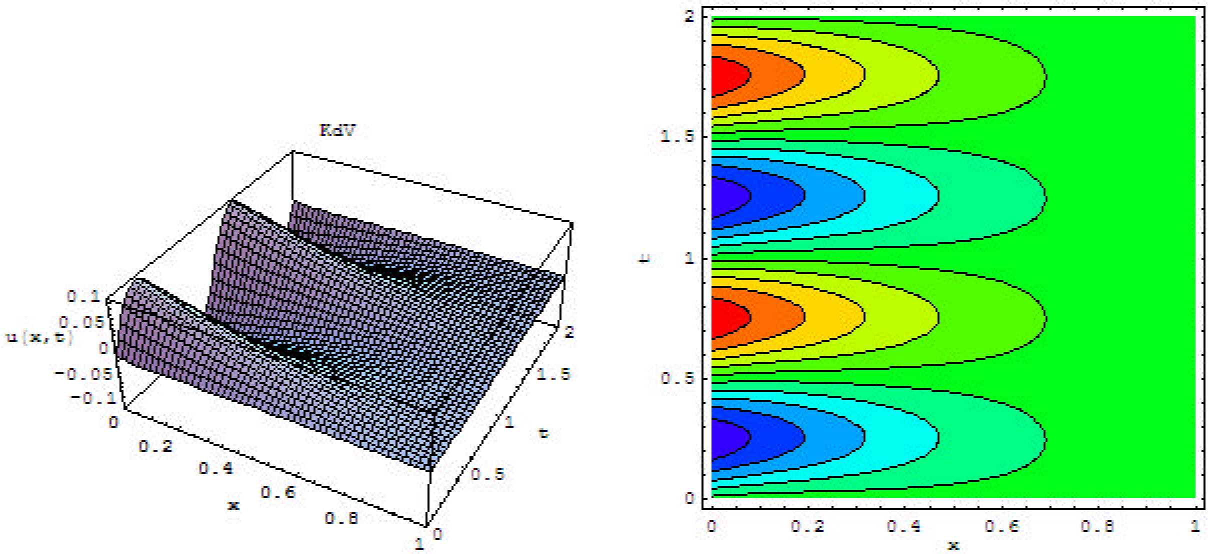

Fig. 1: 3D Plot of the solution (left graph) and contour plot (right graph) with $h(t)$=0.1 sin(2pt)

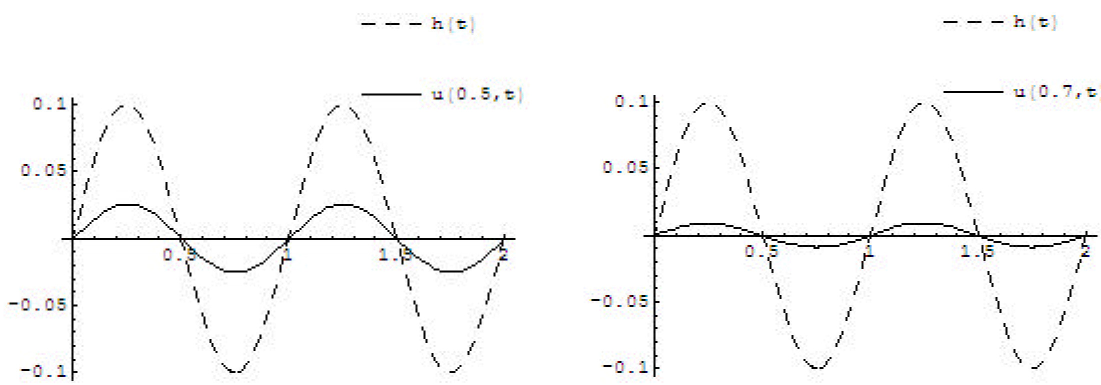

Fig. 2: $u(x, t)$ at $x = 0.5$ (left graph) and at $x = 0.7$ (right graph)

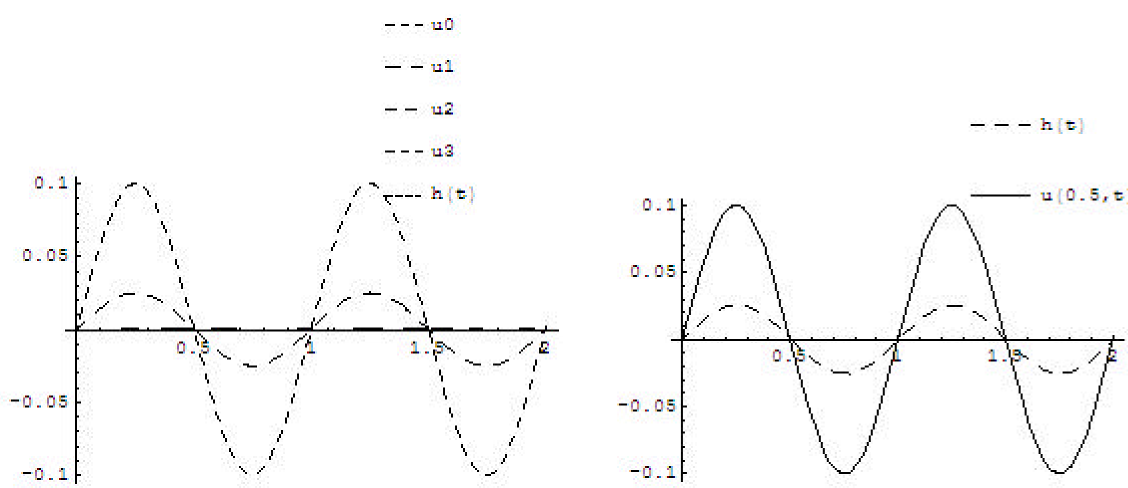

Fig. 3: u_0, u_1, u_2, u_3 and $h(t)$ (left graph) and $u = u_0+u_1+u_2+u_3$ (right graph) at $x = 0.5$

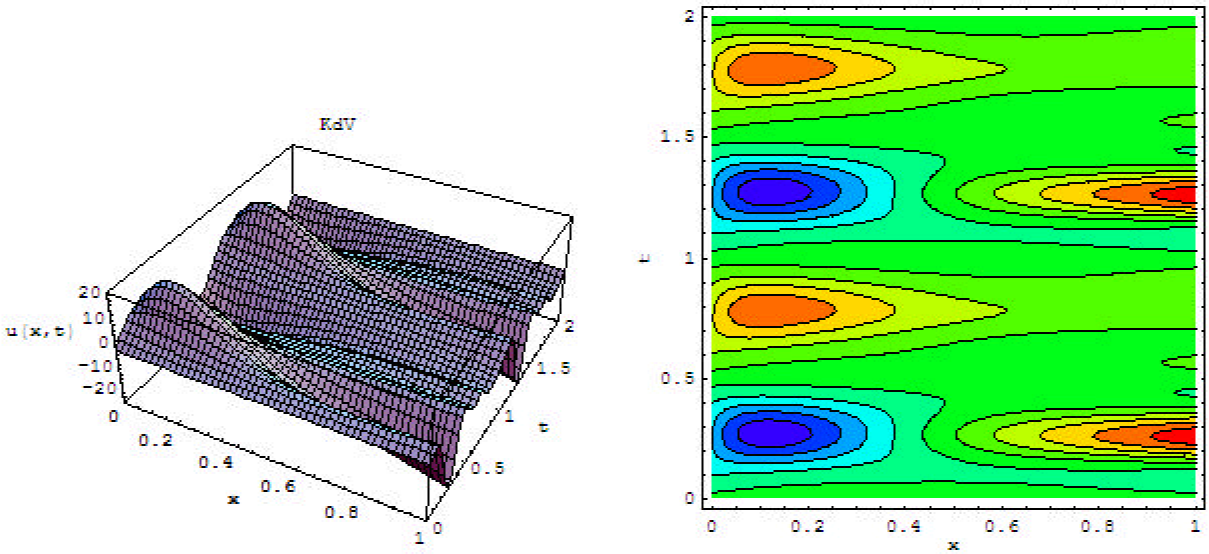

Fig. 4: 3D Plot of the solution (left graph) and contour plot (right graph) with $h(t)$=10 sin(2pt)

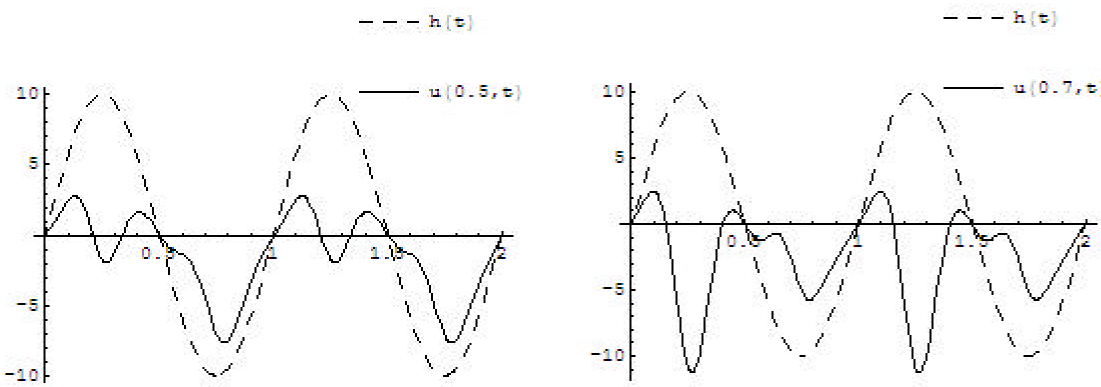

Fig. 5: $u(x, t)$ at $x = 0.5$ (left graph) and at $x = 0.7$ (right graph)

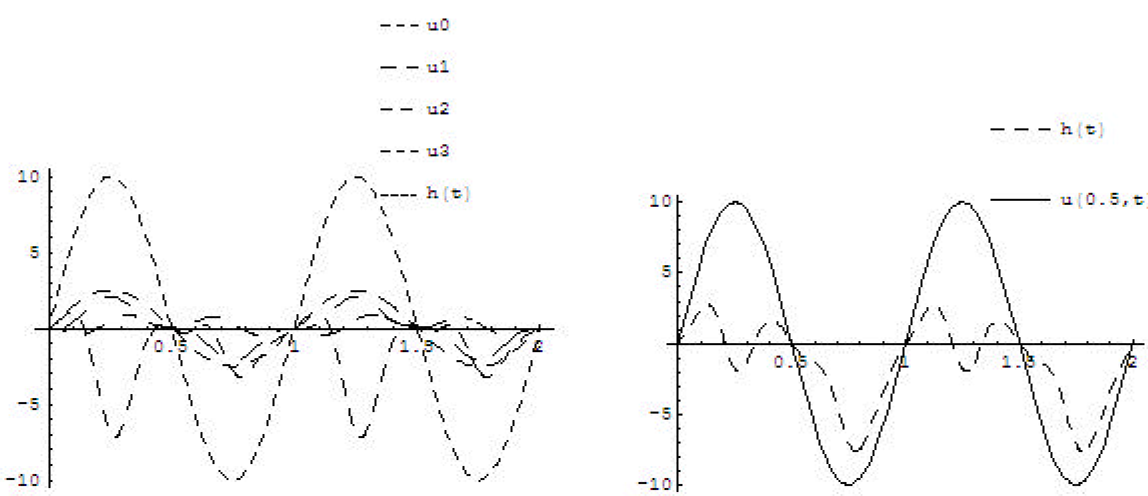

Fig. 6: u_0, u_1, u_2, u_3 and $h(t)$ (left graph) and $u = u_0+u_1+u_2+u_3$ (right graph) at $x = 0.5$

the boundary forcing. There is no restriction imposed on the size of the boundary forcing in this solution.

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