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New Twelfth-Order Modifications of Jarratt's Method for Solving Nonlinear Equations

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Abstract: In this paper we construct new modifications of Jarratt's method for solving nonlinear equations based on the circle of curvature. A precise analysis of convergence is given to show that the new methods are of twelfth-order. Numerical examples are provided to illustrate that the presented methods achieve faster convergence in comparison with other known methods and can compete with the existing methods.

MSC: 41A25 . 65D99

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INTRODUCTION

Many problems in applied mathematics and engineering fields are reduced to finding the solution of a nonlinear equation $f(x) = 0$ and require the employment of an iterative method. In this work we are concerned with iterative methods to find a simple root x^*, i.e., $f(x^*) = 0$ and $f'(x^*)\neq 0$, of a nonlinear equation $f(x) = 0$ that uses f and f' but not the higher derivatives of f. There are many iterative methods such as Newton's method and its variants [1-4], Secant method [5], Halley's method [1-3, 6], Chebyshev method [1-4] and Jarratt's method [7]. Among these methods Newton's method is the most widely used method for the calculation of x^*, which is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

where x_0 is an initial approximation sufficiently close to x^*. It is well known that this method is quadratically convergent [3].

Many researchers developed efficient modifications of existing iterative methods such as those mentioned above in a number of ways to improve their order of convergence at the expense of additional evaluations of functions and/or derivatives mostly at the point iterated by the method, see [8-17] and the references therein. All these modifications are targeted at increasing the order of convergence with a view of increasing the efficiency of the method. Most of these focused on modifications of Newton's method [3, 12, 13, 17] and others on variants of the Chebyshev-Halley methods free from second derivative [10, 11]. Recently, Hou *et al*. [12] presented a new twelfth-order family of methods which improves an eighth-order method [13], a modified Newton's method.

In this paper we are concerned with the construction of new higher order methods which improve Jarratt's method in the order and the efficiency. The Jarratt method [7] is given by

$$x_{n+1} = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)} \tag{2}$$

where

$$J_f(x_n) = \frac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)}$$

and

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}$$

whose order is known to be four. Several efficient modifications of Jarratt's method have been developed and applied in [9, 14-16] to improve the order of convergence. From a practical point of view, it is generally perceived that it is important to improve the order of convergence of this method. We present several higher order modifications of Jarratt's method. By analysis of convergence it is shown that their order of convergence is twelve and we demonstrate by numerical examples their performance in comparison with the other known methods.

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NEW HIGHER ORDER MODIFICATIONS OF JARRATT'S METHOD AND CONVERGENCE ANALYSIS

In this section we develop higher order modifications of Jarratt's method given by (2). Put

$$z_n = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}$$

By an elementary calculation, the circle of curvature at (z_n, $f'(z_n)$) can be found to be

$$\begin{aligned}&\left(x - z_n + \frac{f'(z_n)[1+f'(z_n)^2]}{f''(z_n)}\right)^2 \\ &+\left(y - f(z_n) - \frac{1+f'(z_n)^2}{f''(z_n)}\right)^2 = \frac{(1+f'(z_n)^2)^3}{f''(z_n)^2}\end{aligned} \tag{3}$$

At the intersection point (x_{n+1}, 0) of equation (3) with the x-axis, we get

$$\begin{aligned}&(x_{n+1} - z_n)^2 + 2\frac{f'(z_n)(1+f'(z_n)^2)}{f''(z_n)}(x_{n+1} - z_n) \\ &+ f(z_n)^2 + 2f(z_n)\frac{1+f'(z_n)^2}{f''(z_n)} = 0\end{aligned} \tag{4}$$

Equation (3) can further be rewritten as follows

$$x_{n+1} = z_n - \frac{f(z_n)^2 + 2f(z_n)\dfrac{1+f'(z_n)^2}{f''(z_n)}}{x_{n+1} - z_n + 2\dfrac{f'(z_n)(1+f'(z_n)^2)}{f''(z_n)}} \tag{5}$$

By replacing x_{n+1} on the right-hand side of (5) by the Newton iterate, we obtain the new iterative method

$$x_{n+1} = z_n - \frac{f'(z_n)f''(z_n)f(z_n)^2 + 2f'(z_n)f(z_n)(1+f'(z_n)^2)}{2f'(z_n)^2(1+f'(z_n)^2) - f(z_n)f''(z_n)} \tag{6}$$

We observe that the method (6) requires the evaluation of the second derivative. To derive its second-derivative-free variant, which is important from the practical point of view, let us consider the following approximation

$$f''(z_n) \approx \frac{f'(w_n) - f'(z_n)}{w_n - z_n} \tag{7}$$

where

$$w_n = z_n - \frac{f(z_n)}{f'(z_n)}$$

this giving

$$x_{n+1} = z_n - \frac{f(z_n)[2 + 3f'^2(z_n) - f'(z_n)f'(w_n)]}{f'(z_n) + 2f'^3(z_n) + f'(w_n)} \tag{8}$$

Another new method may be derived by manipulating equation (4) in a different way. Replacing the first term of (4), $(x_{n+1}-z_n)^2$, with $\left(\frac{f(z_n)}{f'(z_n)}\right)^2$ from Newton's iterate (1), results in the following method

$$x_{n+1} = z_n - \frac{f(z_n)^2 f''(z_n) + 2f(z_n)f'(z_n)^2}{2f'(z_n)^3} \tag{9}$$

By using the approximation defined by (7), we obtain the second-derivative-free variant of (9)

$$x_{n+1} = z_n - \frac{1}{2}\left[3 - \frac{f'(w_n)}{f'(z_n)}\right]\frac{f(z_n)}{f'(z_n)} \tag{10}$$

For the method defined by (10), we have the following analysis of convergence. A similar analysis can be done for (6), (8) and (9).

Theorem 2.1 Assume that the function f:D⊂R→R for an open interval D has a simple root x^* in D. If $f(x)$ is sufficiently smooth in a neighborhood of the root x^*, then the method given by (10) is of order twelve.

Proof: Using Taylor expansion and taking into account $f(x^*) = 0$, we have

$$f(x_n) = f'(x^*)[e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots + c_{12}e_n^{12} + O(e_n^{13})] \tag{11}$$

where $e_n = x_n - x^*$ and

$$c_k = \frac{1}{k!}\frac{f^{(k)}(x^*)}{f'(x^*)}, k = 2,3,\cdots$$

and

$$f'(x_n) = f'(x^*)[1 + 2c_2 e_n + 3c_3 e_n^2 + \cdots + 12c_{12}e_n^{11} + O(e_n^{12})] \tag{12}$$

Dividing (11) by (12) gives us

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + \cdots \\ &+ (-1024c_2^{11} + 7936c_2^9 c_3 + \cdots - 11c_{12})e_n^{12} \\ &+ O(e_n^{13})\end{aligned} \tag{13}$$

so that

$$\begin{aligned} y_n - x^* &= \frac{1}{3}e_n + \frac{2}{3}c_2 e_n^2 - \frac{4}{3}(c_2^2 - c_3)e_n^3 + \cdots \\ &\quad + \frac{2}{3}(1024c_2^{11} - 7936c_2^9 c_3 + \cdots + 11c_{12})e_n^{12} + O(e_n^{13}) \end{aligned} \tag{14}$$

By expanding $f'(y_n)$ about x^*, we find

$$f'(y_n) = f'(x^*)\begin{bmatrix} 1 + \frac{2}{3}c_2 e_n + \frac{1}{3}(4c_2^2 + c_3)e_n^2 + \cdots \\ + \frac{4}{27}(9216c_2^{12} + \cdots \\ + 3c_2(-10965c_3^4 c_4 + \cdots + 33c_{12}))e_n^{12} \\ + O(e_n^{13}) \end{bmatrix} \tag{15}$$

From (11), (12) and (15) we therefore have

$$\begin{aligned} J_f(x_n)\frac{f(x_n)}{f'(x_n)} &= e_n - \left(c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^4 + \cdots \\ &\quad + (-496c_2^{11} + 3800c_2^9 c_3 + \cdots - 2c_{12})e_n^{12} \\ &\quad + O(e_n^{13}) \end{aligned} \tag{16}$$

This immediately yields

$$\begin{aligned} z_n - x^* &= \left(c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^4 + \cdots \\ &\quad + (496c_2^{11} - 3800c_2^9 c_3 + \cdots + 2c_{12})e_n^{12} \\ &\quad + O(e_n^{13}) \end{aligned} \tag{17}$$

Expanding $f(z_n)$ and $f'(z_n)$ about x^*, we get

$$f(z_n) = f'(x^*)\begin{bmatrix} (c_2^3 - c_2c_3 + \frac{1}{9}c_4)e_n^4 + \cdots \\ + \frac{1}{11664}(9657792c_2^{11} + \cdots + 23328c_{12})e_n^{12} \\ + O(e_n^{13}) \end{bmatrix} \tag{18}$$

and

$$f'(z_n) = f'(x_n)\begin{bmatrix} 1 + \frac{2}{9}c_2(9c_2^3 - 9c_2c_3 + c_4)e_n^4 + \cdots \\ + (992c_2^{12} + \cdots + 4c_2c_{12})e_n^{12} \\ + O(e_n^{13}) \end{bmatrix} \tag{19}$$

respectively. Dividing (18) by (19) gives us

$$\begin{aligned} \frac{f(z_n)}{f'(z_n)} &= \left(c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^4 + \cdots \\ &\quad + (166c_2^{11} + \cdots + 2c_{12})e_n^{12} + O(e_n^{13}) \end{aligned} \tag{20}$$

Hence we obtain

$$\begin{aligned} w_n - x^* &= \frac{1}{81}c_2(9c_2^3 - 9c_2c_3 + c_4)^2 e_n^8 + \cdots \\ &\quad + \left(330c_2^{11} + \cdots + \frac{2}{9}c_2c_4c_8\right)e_n^{12} + O(e_n^{13}) \end{aligned} \tag{21}$$

Expanding $f'(w_n)$ about x^* produces

$$f'(w_n) = f'(x^*)\begin{bmatrix} 1 + \frac{2}{81}c_2^2(9c_2^3 - 9c_2c_3 + c_4)^2 e_n^8 \\ + \cdots + \left(660c_2^{12} + \cdots + \frac{4}{9}c_2^2c_4c_8\right)e_n^{12} \\ + O(e_n^{13}) \end{bmatrix} \tag{22}$$

It then follows from (17), (19), (20) and (22) that

$$\begin{aligned} e_{n+1} &= z_n - x^* - \frac{1}{2}\left[3 - \frac{f'(w_n)}{f'(z_n)}\right]\frac{f(z_n)}{f'(z_n)} \\ &= \frac{(4c_2^2 + c_3)(9c_2^3 - 9c_2c_3 + c_4)^3}{1458}e_n^{12} + O(e_n^{13}) \end{aligned} \tag{23}$$

This means that the method given by (10) is of twelfth-order.

NUMERICAL EXAMPLES

We present some numerical test results for various iterative methods. The following methods were considered and compared: Newton's method (1) (NM), Jarratt's method (2) (JM), the Hou-Li twelfth-order method [12] (HM) and our new curvature based method (10) (PM).

All computations were done using Maple package with 128 digit floating point arithmetic (Digits:=128). We accept an approximate solution rather than the exact root, depending on the precision (ε) of the computer. The following stopping criteria are used for computer programs: (i)$|x_{n+1}-x_n|<\varepsilon$, (ii) $|f(x_{n+1})|<\varepsilon$ and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root x^* computed. For numerical illustrations in this section we used the fixed stopping criterion $\varepsilon = 10^{-25}$. We used the following test functions and display the computed approximate zeros x^*:

$f_1(x) = x^2 - e^x - 3x + 2,$
$x^* = 0.25753028543986076045536730493724178$

$f_2(x) = \cos x - x,$
$x^* = 0.73908513321516064165531208767387340$

$f_3(x) = x^3 - 10,$
$x^* = 2.1544346900318837217592935665193504 9$

$f_4(x) = e^x + x - 20,$
$x^* = 2.842438953784447067816585940150 95007$

$f_5(x) = (x+2)e^x - 1,$
$x^* = -0.442854401002388583141327999999 33681$

$f_6(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$
$x^* = -1.207647827130918927009416758 4$

$f_7(x) = 2x\cos x + x - 3,$
$x^* = -3.532251691536475964459825850 8$

$f_8(x) = \sqrt{x} - \frac{1}{x} - 3,$
$x^* = 9.633595562832695192406312709 2$

$f_9(x) = \ln x + \sqrt{x} - 5,$
$x^* = 8.309432694231571795346955682 7$

$f_{10}(x) = x^3 + 4x^2 - 10,$
$x^* = 1.365230013414096845760806829 0$

$f_{11}(x) = x^5 + x - 10000,$
$x^* = 6.308777129972689094767571771 8$

The numerical results presented in the tables shown below demonstrate the efficiency of one of the new methods presented in this contribution, which was arbitrarily chosen. Notice that the number of iterations (IT) is given along with the value of the function at the last iteration $f(x_n)$. The new method was superior in most of 11 cases to the Hou-Li method, which is of the same order as ours. Besides, it is observed that our method showed faster convergence than the classical well-known methods. Therefore we conclude that the newly presented methods can be competitive with the other methods known in the literature.

Table 1: f_1, x_0 = 2.0

	NM	JM	HM	PM
$\|x_n-x^*\|$	9.10e-28	4.17e-95	8.34e-82	1.99e-51
$f(x_n)$	2.92e-55	-1.00e-127	-1.00e-127	-1.00e-127
IT	6	5	4	3

Table 2: f_2, x_0 = 1.5

	NM	JM	HM	PM
$\|x_n-x^*\|$	3.19e-32	7.91e-52	1.45e-52	8.20e-118
$f(x_n)$	-3.76e-64	0	0	0
IT	6	4	3	3

Table 3: f_3, x_0 = 4.0

	NM	JM	HM	PM
$\|x_n-x^*\|$	9.17e-37	5.81e-82	2.90e-26	7.11e-41
$f(x_n)$	5.44e-72	0	0	0
IT	8	5	3	3

Table 4: f_4, x_0 = 0.0

	NM	JM	HM	PM
$\|x_n-x^*\|$	8.42e-28	1.56e-69	5.10e-26	2.36e-77
$f(x_n)$	6.08e-54	0	-8.01e+07	0
IT	14	6	52	4

Table 5: f_5, x_0 = 2.0

	NM	JM	HM	PM
$\|x_n-x^*\|$	9.13e-38	1.75e-29	2.28e-54	4.00e-128
$f(x_n)$	9.52e-75	2.18e-116	0	0
IT	10	5	4	4

Table 6: f_6, x_0 =-1.0

	NM	JM	HM	PM
$\|x_n-x^*\|$	8.63e-33	2.39e-50	3.29e-30	4.15e-101
$f(x_n)$	-2.27e-63	-1.10e-126	-1.10e-126	1.20e-126
IT	7	4	3	3

Table 7: f_7, x_0 =-4.8

	NM	JM	HM	PM
$\|x_n-x^*\|$	1.36e-38	1.20e-35	4.45e-87	1.00e-127
$f(x_n)$	-7.55e-76	0	0	0
IT	9	5	6	4

Table 8: f_8, x_0 = 15.5

	NM	JM	HM	PM
$\|x_n-x^*\|$	1.48e-50	1.66e-60	1.04e-49	0
$f(x_n)$	-1.17e-102	0	0	0
IT	7	4	3	3

Table 9: f_9, x_0 = 11.9

	NM	JM	HM	PM
$\|x_n-x^*\|$	1.05e-26	1.73e-66	4.90e-50	0
$f(x_n)$	-1.39e-54	-1.00e-127	-1.00e-127	1.00e-127
IT	6	4	3	3

Table 10: f_{10}, x_0 = 1.6

	NM	JM	HM	PM
$\|x_n-x^*\|$	1.26e-31	2.42e-65	1.02e-60	0
$f(x_n)$	1.28e-61	-6.00e-127	-6.00e-127	-6.00e-127
IT	6	4	3	3

Table 11: f_{11}, $x_0 = 9.8$

	NM	JM	HM	PM
$\lvert x_n-x^*\rvert$	1.37e-48	7.48e-61	2.61e-108	2.02e-28
$f(x_n)$	4.76e-93	0	0	0
IT	9	5	4	3

CONCLUSION

In this paper we presented new efficient twelfth-order modifications of Jarratt's method for solving nonlinear equations, which are based on the circle of curvature and require two function-and four first derivative-evaluations per iteration. From numerical experiments we conclude that for several nonlinear test functions our method shows faster convergence than the other existing methods in comparison and has at least equal performance over those methods.

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