

Application of Homotopy Analysis Method for Water Transport in Unsaturated Porous Media

Hossein Jafari and M.A. Firoozjaee

Department of Mathematics, University of Mazandaran, Babolsar, Iran

Abstract: In this paper, an analytic technique, namely the Homotopy Analysis Method (HAM) has been applied for solving Richards' equation which is one of the most well-known equations to describe the behavior of the infiltration of unsaturated zones in soil as a porous medium. The results show that this method is very efficient and convenient and can be applied to a large class of problems. Comparisons of the results obtained by the HAM with that obtained by the ADM and MADM suggest that both the ADM and MADM are special case of the HAM.

Key words: Homotopy analysis method . richards'equation . burger equation . adomian decomposition method . partial differential equation

INTRODUCTION

Many researchers have been involved in modeling water movement in an unsaturated porous material [4, 7, 8]. Richards [20] derived a governing equation for water flow in soil based on continuum mechanics. In this model, the continuity equation was coupled with Darcy's law as a momentum equation. The following equation is known as the one-dimensional form of Richards equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z} \left(D \frac{\partial u}{\partial z} - K \right) \quad (1)$$

where u is unsaturated soil moisture content, K is conductivity and D is soil water diffusivity. The sophistication of the analytical and numerical methods that are available to solve the governing equation of unsaturated flow in soils (Richards equation) makes it necessary to have suitable models for parameters in the equation. Several methods are available for the estimation of such parameters, such as conductivity and water diffusivity [21]. Basically, there are three commonly used models: (i) Brook-Coreys model [3, 6], (ii) the van Genuchten model and (iii) the exponential model. The Brooks-Corey model introduces a well-defined air-entry value that is associated with the largest pore size, assuming complete wet ability Brooks-Corey model soils can be simplified to the following equations by some further considerations [6, 26]:

$$K(u) = K_0 u^k \quad \text{form } \geq 1 \quad (2)$$

$$D(u) = D_0(n+1)u^n \quad \text{form } \geq 1 \quad (3)$$

where K_0 , D_0 , n and k are constants representing soil properties such as pore-size distribution, particle shape, etc. In these relations, u is scaled between 0 and 1 and the form of diffusivity is normalized so that

$$\int_0^1 D(u) du = 1 \quad \forall n$$

Most of the results in this paper can be applied to the general values of k and n . It is believed that the Brook-Corey model is widely used because of its well-defined configuration. There are several analytical and numerical solutions to Richards equation considering the Brook-Corey model. The choice $n = 0$ and $k = 0$ in Eqs. (2) and (3) yields the classic Burgers equations. For general values of k and n , the generalized Burgers equation, which was the focus of several researchers including [11], is obtained. In the special case $n = 0$, Richards equation reduces to a linear equation. Richards equation with other models was also solved numerically and by various innovative and common analytical methods. Some of these solutions are limited to very simple geometrical and initial conditions. In the last 30 years many finite difference and finite element numerical solutions were developed, even in 2D and 3D. Another numerical method, the finite volume method, looks quite promising for solving Richards equation, especially when sharp infiltration fronts develop and must be approximated on unstructured

multidimensional grids. For instance, a few researchers have tried to solve the equation by the Adomian decomposition method (ADM) as an analytical series solution [22-24]. In this study, a simplified Brooks-Corey model (Eqs. (2) and (3)) was applied on Richards' equation [Eq. (1)]. However, two cases for conductivity exponents (with equal soil water diffusivity) and the solutions were tried. The hyperbolic tangent function is commonly applied to solve these transform equations [25]. The general form of Burgers equation in the order of (n,1) is:

$$u_t + a(u^n)_x + bu_{xx} = 0, n \geq 1, a, b \neq 1 \quad (4)$$

and its exact solution is

$$u(x,t) = \frac{c}{2a} (1 + \tanh[\frac{c(n-1)}{2b}(x-ct)])^{\frac{1}{n-1}} \quad (5)$$

In these cases, Richard equation coincides with Burger equation (n,1), where u is water content, x is depth (cm) and c is an arbitrary coefficient. In 1992, Liao [16] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy Analysis Method (HAM), [16-18]. This method is successfully applied to solve many types of nonlinear problems [2, 9, 10, 14, 15, 19].

BASIC IDEA OF HAM

We consider the following differential equation

$$N[u(\tau)] = 0 \quad (6)$$

where N is a nonlinear operator, τ denotes independent variable, u (τ) is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [17] constructs the so-called zero-order deformation equation

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] = pH(\tau)N[\phi(\tau;p)] \quad (7)$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, H(τ) is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(\tau;0) = u_0(\tau), \phi(\tau;1) = u(\tau) \quad (8)$$

respectively. Thus, as p increases from 0 to 1, the solution $\phi(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau;p)$ in Taylor series with respect to p, we have

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m \quad (9)$$

Where

$$u_m(\tau) = \frac{1}{m!} \left. \frac{\partial^m \phi(\tau;p)}{\partial p^m} \right|_{p=0} \quad (10)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function are properly chosen, the series (9) converges at $p = 1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) \quad (11)$$

which must be one of solutions of original nonlinear equation, as proved by liao [17]. As $h = -1$ and $H(\tau) = 1$, Eq. (7) become

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] + pN[\phi(\tau;p)] = 0 \quad (12)$$

Which is used mostly in the homotopy perturbation method, where as the solution obtained directly, without using Taylor series [12, 13]. According to the definition (10), the governing equation can be deduced from the zero-order deformation equation (7).

Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \dots, u_n(\tau)\}$$

Differentiating equation (7) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = hH(\tau) R_m(\vec{u}_{m-1}) \quad (13)$$

subject to initial condition

$$u_m(x,0) = 0, \frac{\partial u_m(x,0)}{\partial t} = 0 \quad (14)$$

Where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(\tau, p)]}{\partial p^{m-1}} \right|_{p=0} \quad (15)$$

And

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases} \quad (16)$$

It should be emphasized that $u_m(\tau)$ for $m \geq 1$ is governed by the linear equation (13) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as *Mathematica*. If Eq. (6) admits unique solution, then this method will produce the unique solution. If equation (6) does not possess unique solution, the HAM will give a solution among many other (possible) solutions. For the convergence of the above method we refer the reader to Liao's work [17].

APPLICATIONS

In this section we apply HAM for solving Richards equation. Further we compare the gained result with both exact and MADM.

Case 1: In this case a conductivity term is selected as a function of cubic water content and constant

$$K = \frac{u^3}{3} \text{ cm/h}$$

and $D = 1 \text{ cm}^{2/h}$. Therefore, Richard equation becomes:

$$u_t + u^2 u_x - u_{xx} = 0 \quad (17)$$

with initial condition:

$$u(x, 0) = \sqrt{\frac{1}{2} (1 + \tanh[-\frac{x}{3}])}$$

Therefore in Eq.(4) $a = 1/3$, $n=3, b=1$ and we choose $c = 1/3$. For application of homotopy analysis method, we choose the initial approximation:

$$u_0(x, t) = u(x, 0) = \sqrt{\frac{1}{2} (1 + \tanh[-\frac{x}{3}])} \quad (18)$$

and the linear operator:

$$L[\phi(x, t; p)] = \frac{\partial \phi(x, t; p)}{\partial t} \quad (19)$$

which possesses the property

$$L(c_1) = 0 \quad (20)$$

where c_1 is an integral constant to be determined by initial condition. Using Eq. (17), we define nonlinear operator as

$$N[(\partial \phi(x, t; p))] = \frac{\partial \phi(x, t; p)}{\partial t} - \frac{\partial^2 \phi(x, t; p)}{\partial x^2} + \phi^2(x, t; p) \frac{\partial \phi(x, t; p)}{\partial x}$$

In view of Eq. (15), we obtained

$$R_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}(x, t)}{\partial t} - \frac{\partial^2 u_{m-1}(x, t)}{\partial x^2} + \sum_{n=0}^{m-1} \frac{\partial u_{m-1-n}(x, t)}{\partial t} u_j(x, t)$$

Now using Eq.(13), The HAM would lead to:

$$\begin{aligned} u_1(x, t) &= \frac{ht \text{sech}^2(\frac{x}{3}) \tanh(\frac{x}{3})}{18\sqrt{2}(1 - \tanh(\frac{x}{3}))^{\frac{3}{2}}} - \frac{ht \text{sech}^2(\frac{x}{3})}{18\sqrt{2}(1 - \tanh(\frac{x}{3}))^{\frac{3}{2}}} \\ u_2(x, t) &= \frac{h^2 t^2 \text{sech}^2(\frac{x}{3}) \tanh(\frac{x}{3})}{216\sqrt{(2 - 2\tanh(\frac{x}{3}))}} + \frac{ht \tanh(\frac{x}{3}) \text{sech}^2(\frac{x}{3})}{18\sqrt{2}(1 - \tanh(\frac{x}{3}))^{\frac{3}{2}}} \\ &\quad - \frac{h^2 t^2 \text{sech}^2(\frac{x}{3})}{648\sqrt{(2 - 2\tanh(\frac{x}{3}))}} - \frac{h^2 t \text{sech}^2(\frac{x}{3})}{18\sqrt{(2 - 2\tanh(\frac{x}{3}))}} \\ &\quad - \frac{ht \text{sech}^2(\frac{x}{3})}{18\sqrt{2}(1 - \tanh(\frac{x}{3}))^{\frac{3}{2}}} \\ &\quad \vdots \end{aligned}$$

Thus

$$u(x, t) = \sum_{n=0}^{+\infty} u_n(x, t)$$

Figure 1 and 2 show the 5th-order approximate solution HAM and exact solution, respectively.

The behavior of the HAM and the exact solution and MADM in two transition points ($x = 1, 5$) are plotted versus time in Fig. 3 and 4. In [27] this case has been solved by MADM using HAM for $h = -0.85$

Case 2: In this case a conductivity term is selected as a function of quadric water content and constant value

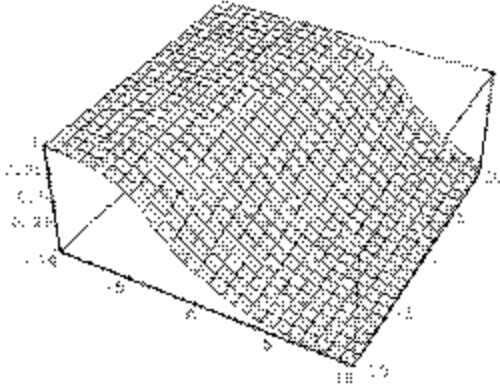


Fig. 1: The 5th-order approximation

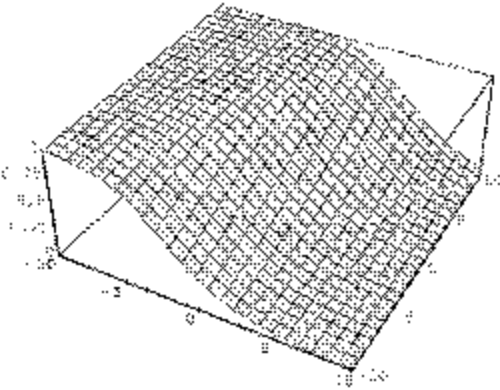


Fig. 2: Exact solution of burger (2/1)

$$K = \frac{u^4}{4} \text{ cm/h}$$

and $D = 1 \text{ cm}^{2/\text{h}}$ Eq.(1). Consequently, Richards equation is converted to:

$$u_t + u^3 u_x - u_{xx} = 0 \quad (21)$$

with initial condition:

$$u(x,0) = \sqrt[3]{\frac{1}{2}(1 + \tanh[-\frac{3}{8}x])}$$

Therefore, in Eq.(4) $a = 1/4$, $n = 4$, $b = 1$ and we choose $c = 1/4$. We choose the linear operator:

$$L[\phi(x,t;p)] = \frac{\partial \phi(x,t;p)}{\partial t} \quad (22)$$

which possesses the property:

$$L(c_1) = 0 \quad (23)$$

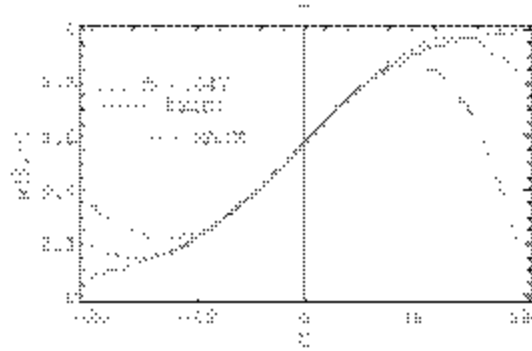


Fig. 3: Comparison of exact solution

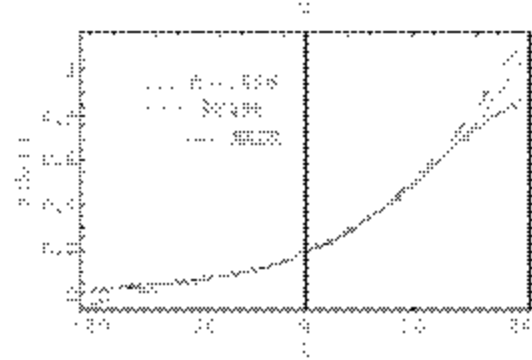


Fig. 4: Comparison of exact solution and 5th-order of HAM and MADM and 5th-order of HAM and MADM

We choose

$$u_0(x,t) = u(x,0) = \sqrt[3]{\frac{1}{2}(1 + \tanh[-\frac{3}{8}x])}$$

From Eq. (21) the nonlinear operator is defined as

$$N[(\phi(x,t;p))] = \frac{\partial \phi(x,t;p)}{\partial t} - \frac{\partial^2 \phi(x,t;p)}{\partial x^2} + \phi^3(x,t;p) \frac{\partial \phi(x,t;p)}{\partial x}$$

Then by application Eq. (15), we have:

$$\begin{aligned} R_m(u_{m-1}) &= \frac{\partial u_{m-1}(x,t)}{\partial t} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} \\ &+ \sum_{n=0}^{m-1} \frac{\partial u_{m-1-n}(x,t)}{\partial t} \sum_{j=0}^n u_{j-k}(x,t) \sum_{k=0}^j u_k(x,t) \end{aligned}$$

Therefore by using Eq. (13), we obtain terms of HAM:

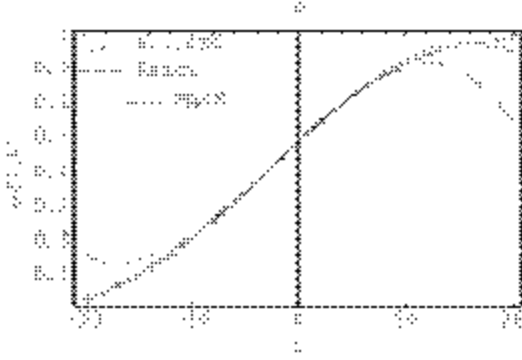


Fig. 5: Comparison of exact solution

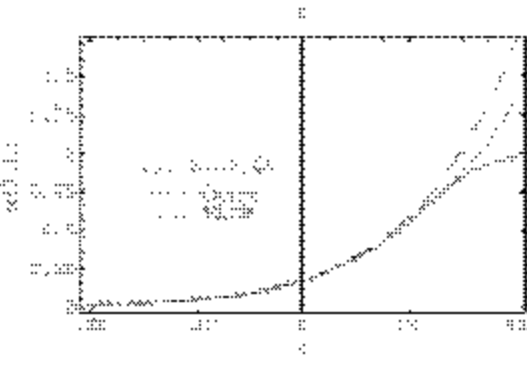


Fig. 6: Comparison of exact solution and 4th-order of HAM and MADM and 4th-order of HAM and MADM

$$u_1(x,t) = \frac{ht \operatorname{sech}^2\left(\frac{3x}{8}\right) \tanh\left(\frac{3x}{8}\right)}{32\sqrt[3]{2}\left(1 - \tanh\left(\frac{3x}{8}\right)\right)^{\frac{5}{3}}} - \frac{ht \operatorname{sech}^2\left(\frac{3x}{8}\right)}{32\sqrt[3]{2}\left(1 - \tanh\left(\frac{3x}{8}\right)\right)^{\frac{5}{3}}}$$

$$u_2(x,t) = -\frac{5h^2 t^2 \operatorname{sech}^6\left(\frac{3x}{8}\right) \tanh\left(\frac{3x}{8}\right)}{512\sqrt[3]{2}\left(1 - \tanh\left(\frac{3x}{8}\right)\right)^{\frac{11}{3}}} - \frac{15h^2 t^2 \operatorname{sech}^6\left(\frac{3x}{8}\right)}{2048\sqrt[3]{2}\left(1 - \tanh\left(\frac{3x}{8}\right)\right)^{\frac{8}{3}}}$$

$$+ \frac{5h^2 t^2 \operatorname{sech}^6\left(\frac{3x}{8}\right)}{512\sqrt[3]{2}\left(1 - \tanh\left(\frac{3x}{8}\right)\right)^{\frac{11}{3}}} + \frac{35h^2 t^2 \tanh^2\left(\frac{3x}{8}\right) \operatorname{sech}^4\left(\frac{3x}{8}\right)}{2048\sqrt[3]{2}\left(1 - \tanh\left(\frac{3x}{8}\right)\right)^{\frac{8}{3}}} - \dots$$

$$\vdots$$

Thus

$$u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t)$$

Now, Richards equation is approximated by repeating the HAM. The behavior the 4th order of HAM and the exact solution MADM in two transition points ($x = 1; 7$) are plotted versus time in Fig. 5 and 6. By choosing $h = -1$, we will reach ADM solution given by [17].

Remark: In Fig. 3-6, we show the comparisons between the 4-term HAM solutions and the exact solutions. We observe that the results of the 4-term HAM are very close to the exact solutions which confirm the validity of the HAM.

All the numerical results obtained by the 4-term HAM are exactly same as the ADM solutions and HPM solutions for special case $h = -1$, $H(t) = 1$. So it means that the ADM is a special case of HAM. But HAM is more general and contains the auxiliary parameter h , which provides us with a simple way to adjust and control the convergence region of solution series. As pointed out by Abbasbandy in [1] one had to choose a proper value of h to ensure the convergence of series solution for strongly nonlinear problems.

So it is more important to show that the HAM gives convergent series solution for any larger values of t by choosing proper values of h .

CONCLUSION

In this paper, we have successfully developed HAM for solving model Broke-Corey in Richards equation for all values of x and various intervals of t .

It is apparently seen that HAM is very powerful and efficient technique in finding analytical solutions for wide classes of nonlinear problems.

It is worth pointing out that this method presents a rapid convergence for the solutions. HAM provides accurate numerical solution for nonlinear

problems in comparison with other methods such as MADM. They also do not require large computer memory and discrimination of the variables t and x . The results show the validity and great potential of the homotopy analysis method for nonlinear problems in science and engineering.

Mathematica has been used for computations in this paper.

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