Studies in Nonlinear Sciences 1 (1): 01-07, 2010
ISSN 2221-3910

The Boubaker Polynomials Expansion Scheme for Solving Applied-physics Nonlinear high-order Differential Equations

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Abstract: In this study, the main features of the Boubaker Polynomials Expansion Scheme (BPES) are outlined on the basis of published, compared and confirmed solution to differently established applied-physics nonlinear problems. Some new applications of the same scheme in the field of fluids motion and waves dynamics are also discussed.

PACS: 2010 02.30.Jr; 02.30.Sa

INTRODUCTION

In the last decades, many solutions of nonlinear physics problems have been proposed using sequences and polynomial expansions [1-5]. An attempt to solve a nonlinear heat transfer problem has been carried out in the context of the enhancement of the spray pyrolysis common setup [6]. A procedure step during this attempt yielded the Boubaker polynomials through a proposed numerical expansion. An extracted sequence of these polynomials led to the definition of the Boubaker Polynomials Expansion Scheme (BPES) whose main feature consisted, in this particular case, of involving the boundary conditions in the main heat equation.

In this paper, we present the chronological history of the Boubaker Polynomials Expansion Scheme BPES along with some recent original applications. We particularly investigate the Boubaker Polynomials Expansion Scheme (BPES) analytical features and simplifying performance.

HISTORICAL PREVIEW

The early attempt to solve the heat equation: In the model of the spray pyrolysis common setup, the heat transfer equation was established as equation (1).

$$\begin{cases} \dfrac{\partial^2 T_g(z,t)}{\partial z^2} = \dfrac{1}{D_g}\dfrac{\partial T_g(z,t)}{\partial t} - \dfrac{1}{k_g}.(P_b - P_s) \\ \dfrac{\partial^2 T_s(z,t)}{\partial z^2} = \dfrac{1}{D_s}\dfrac{\partial T_s(z,t)}{\partial t} - \dfrac{1}{k_s}.P_s \end{cases} \qquad (1)$$

with:

T_g : Absolute temperature inside glass medium (in K)
T_s : Absolute temperature inside deposited layer (in K)
D_g : Glass medium thermal diffusivity (in $m^2.s^{-1}$)
D_s : Deposited layer thermal diffusivity (in $m^2.s^{-1}$)
P_b : Power transmitted from bulk to glass (in Wm^{-3})
P_s : Power transmitted from glass to layer (in Wm^{-3})
k_g : Glass medium thermal conductivity (in $W.m^{-1}.K^{-1}$)
k_s : Deposited layer thermal conductivity (in $W.m^{-1}.K^{-1}$)

Under some physical assumptions (ambient fluid weak thermal conductivity, bulk considered as an infinite source under constant temperature T_b…), the Dirichlet-Newman first order boundary conditions were:

$$\begin{cases} T_g(z,t)\big|_{z=0} = T_s(z,t)\big|_{z=0} \\ T_g(z,t)\big|_{z=-H} = T_b \\ k_g.\dfrac{\partial T_g(z,t)}{\partial z}\Big|_{z=0} = k_s.\dfrac{\partial T_s(z,t)}{\partial z}\Big|_{z=0} \approx k_s\dfrac{T_\infty - T_b}{h} \\ k_g.\dfrac{\partial T_g(z,t)}{\partial z}\Big|_{z=-H} = 0 \end{cases} \qquad (2)$$

with:

T_8 : Room absolute temperature (in K)
T_b : Bulk absolute temperature (in K)
h : Deposited layer width (in m)

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For resolution purposes, a general expression (3) was proposed for normalized temperature distribution inside glass sample:

$$T_n(z,t) = \frac{1}{N} e^{-\frac{A}{\frac{H}{z}+1}} \sum_{m=0}^{N} \xi_m . J_m(t) \quad \text{for:} - H < z < 0 \qquad (3)$$

where J_m is m-order first kind Bessel function, N is prefixed integer parameter, A and ξ_m are constants to be found.

By introducing the expression (3) in equation (1) and thanks to Bessel functions expansion uniqueness the system (4) was obtained.

$$\begin{cases} Q_1(z)\times\xi_0 = \xi_1 \\ Q_1(z)\times\xi_1 = -2\xi_0 + \xi_2 \\ Q_1(z)\times\xi_m = \xi_{m-1} + \xi_{m+1} \quad \text{for: } 1 < m < N \\ \ldots \\ Q_1(z)\times\xi_{N-1} = \xi_{N-2} + \xi_N \\ \left.\begin{array}{l} \xi_{N+1} = \xi_{N+3} = \xi_{N+5} = \ldots = \xi_{N+2k+1} = 0; \\ \xi_{N+2} = \xi_{N+4} = \xi_{N+6} = \ldots = \xi_{N+2k} = -(1)^k \xi_N \end{array}\right\} k > 0 \end{cases} \qquad (4)$$

Finally, coefficients ξ_m were calculated for z=0 and for the given values of the parameters H, h, k_g, k_s, D_g and D_s. The calculated set of values, when introduced in expression (3) gave a t-dependent solution to the heat equation.

For superior values of N and when z=0; the system (4) defined a serial of polynomial function $B_m(x)$ as equation (5):

$$\begin{cases} B_0(X) = 1 \\ B_1(X) = X \\ B_2(X) = X^2 + 2 \\ B_m(X) = X.B_{m-1}(X) - B_{m-2}(X) \quad \text{for: } m > 2 \end{cases} \qquad (5)$$

where

$$X = Q_1(z)\big|_{z=0}$$

Later, a monomial definition of these polynomials was established by Labiadh *et al*. [7]:

$$B_n(X) = \sum_{p=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right] . (-1)^p . X^{n-2p} \qquad (6)$$

where:

$$C_{n-p}^{p} = \frac{(n-p)!}{p!(n-2p)!}$$

and

$$\xi(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + \left((-1)^n - 1 \right)}{4}$$

($\lfloor \ \rfloor$ denotes the floor function).

Recently, Luzón *et al*. [8] and Barry *et al*. [9] established Riordan matrices for the Boubaker polynomials:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\ -2 & 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & -3 & 0 & -2 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (7)$$

and

$$\left(\begin{array}{c|cccccccc} \frac{1}{3} & & & & & & & & \\ 0 & 1 & & & & & & & \\ 0 & 0 & 3 & & & & & & \\ 0 & -2 & 0 & 9 & & & & & \\ 0 & 0 & -12 & 0 & 27 & & & & \\ 0 & 4 & 0 & -54 & 0 & 81 & & & \\ 0 & 0 & 36 & 0 & -216 & 0 & 243 & & \\ 0 & -8 & 0 & 216 & 0 & -810 & 0 & 729 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \qquad (8)$$

They also demonstrated [8, 9] that the Boubaker polynomials were linked to the Fermat polynomials $F_n(x)$ by the relation:

$$B_n(x) = \frac{1}{\left(\sqrt{2}\right)^n} F_n\left(\frac{2\sqrt{2}\times x}{3}\right) + \frac{1}{\left(\sqrt{2}\right)^{n-2}} F_{n-2}\left(\frac{2\sqrt{2}\times x}{3}\right); n = 0,1,2,\ldots \qquad (9)$$

Boubaker Polynomials Expansion Scheme (BPES): Recently, it was noticed some particularities for the orders m=4q, (q>0) for the Boubaker polynomials. These particularities led to the definition of the polynomial sub-set B_{4q}:

$$\begin{cases} B_0(X) = 1; \\ B_4(X) = X^4 - 2; \\ B_8(X) = X^8 - 4X^6 + 8X^2 - 2; \\ B_{12}(X) = X^{12} - 8X^{10} + 18X^8 - 35X^4 + 24X^2 - 2; \\ B_{16}(X) = X^{16} - 12X^{14} + 52X^{12} - 88X^{10} \\ \qquad + 168X^6 - 168X^4 - 48X^2 - 2; \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ B_{4(q+1)}(X) = (X^4 - 4X^2 + 2)\times B_{4(q)}(X) - B_{4(q-1)}(X) \end{cases} \qquad (10)$$

The first noticed properties of the 4q-Boubaker polynomials were the following:

Values at boundaries, in the reduced real domain $[0; \alpha_q]$:

$$\begin{cases} \left.\sum_{q=1}^{N} B_{4q}(x)\right|_{x=0} = -2N \neq 0; \\ \left.\sum_{q=1}^{N} B_{4q}(x)\right|_{x=\alpha_q} = 0; \end{cases} \tag{11}$$

first derivatives values at boundaries:

$$\begin{cases} \left.\sum_{q=1}^{N} \frac{dB_{4q}(x)}{dx}\right|_{x=0} = 0 \\ \left.\sum_{q=1}^{N} \frac{dB_{4q}(x)}{dx}\right|_{x=\alpha_q} = \sum_{q=1}^{N} H_q \end{cases} \tag{12}$$

with:

$$H_n = B'_{4n}(\alpha_n) = \left(\frac{4\alpha_n [2-\alpha_n^2] \times \sum_{q=1}^{n} B_{4q}^2(\alpha_n)}{B_{4(n+1)}(\alpha_n)} + 4\alpha_n^3 \right)$$

and: second derivatives values at boundaries:

$$\begin{cases} \left.\sum_{q=1}^{N} \frac{d^2 B_{4q}(x)}{dx^2}\right|_{x=0} = \frac{8}{3}\left(N(N^2-1)\right) \\ \left.\sum_{q=1}^{N} \frac{d^2 B_{4q}(x)}{dx^2}\right|_{x=\alpha_q} = \sum_{q=1}^{N} G_q \end{cases} \tag{13}$$

with:

$$G_q = \left.\frac{d^2 B_{4q}(x)}{dx^2}\right|_{x=\alpha_q} = \frac{3\alpha_q(4q\alpha_q^2+12q-2)H_q - 8q(24q^2\alpha_q^2+8q^2-3q+4)}{(\alpha_q^2-1)(12q\alpha_q^2+4q-2)}$$

where α_q (or α_n) designates the 4q-Boubaker polynomial minimal positive root.

Equations (11-13) were the first bases for the establishment of the Boubaker Polynomials Expansion Scheme (BPES).

Solutions to several applied physics problems have been proposed [10-15] using the (BPES).

The Boubaker Polynomials Expansion Scheme (BPES) applications

A glossary of precedent applications: The Boubaker Polynomials Expansion Scheme BPES has been used by several applied physics and engineering studies. Agida *et al*. [10] used this protocol for establishing an analytical method for solving Love's integral equation in the case of a rational kernel. O. B. Awojoyogbe *et al*. [11] took profit from the similarities between the hemodynamic flow system inside some organic tissues and the BPES definition system, in order to express the tissue response to magnetic fields excitation. Kumar [12] combined the Boubaker Polynomials Expansion Scheme (BPES) analyses and array analyses for determining the normalized field created conjointly by two similar circular coaxial conducting disks separated by a pre-fixed distance. On an other hand, J. Ghanouchi *et al*. [13] used the BPES to discuss the intriguing paradox of establishment of non-Gaussian isothermal generative lines beneath a plate surface targeted by a Gaussian beam.

The works carried out by S. Slama *et al*. [14, 15] proposed solutions to the heat transfer problem inside different welding and annealing systems. These works used the BPES as a guide to solve heat discrete conservation equations during cooling phases and yielded consistent cooling velocities profile. T. Ghrib *et al*. [16] used the BPES in order to establish a first order correlation between the Vickers microhardness and the thermal diffusivity of treated steel alloys.

In the last years, S. Lazzez *et al*. [17, 18] and D.H. Zhang *et al*. [19] investigated semiconductor micro layers physical properties using the BPES. More recently, A. Yildirim *et al*. [20] proposed analytical solutions to the Klein-Gordon equation in a pulsed stationary regime using Modified Variational Iteration Method (MVIM) and BPES. Geng's standard second-order Boundary Value Problem (BVP) was also investigated by D.H. Zang *et al*. [21] using the BPES.

In a different filed like animal biology and medical sciences, Dubey *et al*. [22] proposed, in a study commented and corrected by Milgram [23], an analytical method for the identification of predator-prey populations time-dependent evolution in a Lotka-Volterra predator-prey model which took into account the concept of accelerated-predator-satiety, O. B. Awojoyogbe *et al*. [24] proposed also a mathematical formulation for the NMR diffusion equation derived from the Bloch NMR flow in lower heart coronary artery. This formulation was totally based on the properties of the Boubaker polynomials expansion scheme BPES.

Application to the steady-state stagnation point flow problem: The steady-state boundary-layer flow of a micro-polar fluid near the forward stagnation point of an infinite flat plane [25] is governed by the system:

$$\begin{cases}(1+H)\dfrac{d^3 f(\eta)}{d\eta^3}+H\dfrac{dg(\eta)}{d\eta}+\dfrac{d^2 f(\eta)}{d\eta^2}-\left(\dfrac{df(\eta)}{d\eta}\right)^2=-1\\ (1+\dfrac{H}{2})\dfrac{d^2g(\eta)}{d\eta^2}+f(\eta)\dfrac{dg(\eta)}{d\eta}-g(\eta)\dfrac{df(\eta)}{d\eta}-H\left(2g(\eta)+\dfrac{d^2f(\eta)}{d\eta^2}\right)=0\\ f(\eta)\Big|_{\eta=0}=\dfrac{df(\eta)}{d\eta}\Big|_{\eta=0}=\dfrac{d^2g(\eta)}{d\eta^2}\Big|_{\eta=0}=0\\ \lim\limits_{\eta\to+\infty}\dfrac{df(\eta)}{d\eta}=1\\ \lim\limits_{\eta\to+\infty}g(\eta)=0\end{cases} \tag{14}$$

where H is the vortex-to-absolute viscosities ratio. The dimensionless functions $f(\eta)$ and $g(\eta)$ are expressed versus stream function $\psi(x,y)$ and the gyration component N(x,y):

$$\begin{cases}\psi(x,y)=x\sqrt{a\upsilon}f(\eta)\\ N(x,y)=x\sqrt{a^3\upsilon^{-1}}g(\eta)\end{cases} \tag{15}$$

where a is a constant, υ is the kinematic viscosity in the case of concentrated particle flows in which the microelements close to the wall surface are unable to rotate and (x, y) are the cartesian coordinates with x-axis being along the wall and the y-axis normal to it.

Classical analytical resolution protocols are based, i. e., on affecting exponentially decaying forms to the functions $f(\eta)$ and $g(\eta)$:

$$\begin{cases}f(\eta)=\displaystyle\sum_{i=0}^{+\infty}\sum_{j=1}^{+\infty}A_{ij}\eta^i e^{-n\lambda\eta}\\ g(\eta)=\displaystyle\sum_{i=0}^{+\infty}\sum_{j=1}^{+\infty}B_{ij}\eta^i e^{-n\lambda\eta}\end{cases} \tag{16}$$

where n and λ are arbitrary constants and (A_{ij}, B_{ij}) are couples of unknown indexed coefficients.

Initial guesses are generally applied to the expressions of $f(\eta)$ and $g(\eta)$ parallel to the establishment of appropriated operators:

$$\begin{cases}\Im_f=\dfrac{d^3F}{d\eta^3}-\lambda^2\dfrac{dF}{d\eta}\\ \Im_g=\dfrac{d^2G}{d\eta^2}-\lambda^2 G\end{cases} \tag{17}$$

The BPEs resolution protocol is fundamentally different since its proposed expansions (18) verify the boundary conditions intrinsically, inherently and besides all prior to resolution process:

$$\begin{cases}f(\eta)=\dfrac{1}{2N_0}\displaystyle\sum_{k=1}^{N_0}\lambda_k\times B_{4k}(\eta\times r_k)\\ g(\eta)=\dfrac{1}{2N_0}\displaystyle\sum_{k=1}^{N_0}\tilde{\lambda}_k\times B_{4k}(\eta\times r_k)\end{cases} \tag{18}$$

where B_{4k} are the 4k-order Boubaker polynomials, r_k are B_{4k} minimal positive roots, N_0 is a prefixed integer and $\lambda_k,\tilde{\lambda}_k\Big|_{k=1..N_0}$ are unknown pondering real coefficients.

Application of the BPES needs no additional operator. Its following step consists simply of transforming the main system (14) into a nonlinear problem based on minimizing Minimum Square Functions

$$\Psi_{MS}(\lambda_k\Big|_{k=1..N_0},\tilde{\lambda}_k\Big|_{k=1..N_0}):$$

$$\Psi_{MS}(\lambda_k\Big|_{k=1..N_0},\tilde{\lambda}_k\Big|_{k=1..N_0})=\left(\sum_{k=1}^{N_0}\Gamma(\lambda_k)-\sum_{k=1}^{N_0}\tilde{\Gamma}(\tilde{\lambda}_k)\right)^2 \tag{19}$$

where Γ and $\tilde{\Gamma}$ are nonlinear function deduced from system (14).

The correspondent solutions are represented in Fig. 1 along with the exact solutions given by H. Xu [25].

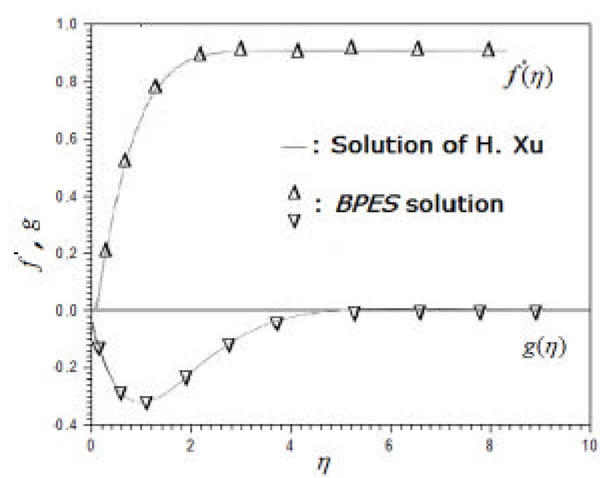

Fig. 1: Proposed solution to the steady-state stagnation point flow problem

Application to the problem of flat plate impulsive motion: This problem concerns second-order fluids [26] unidirectional flow at the vicinity of a moving flat plate. If the plate is situated at a given spacial position (wall: y = 0) and by denoting ϖ the velocity component along the wall, the governing equation is:

$$\begin{cases} \dfrac{\partial\varpi}{dt} = k\dfrac{\partial^2\varpi}{dy^2} + l\dfrac{\partial^3\varpi}{dy^2dt} \\ t \geq 0; y \geq 0 \end{cases} \tag{20}$$

where k is kinematic viscosity [27, 28] and l is of the fluid stress-to-density ratio [27]. It is noted that the case l = 0 is simply referring to a Newtonian flow [26].

The solution is subjected to the standard physically imposed boundary conditions:

$$\begin{cases} \varpi(0,y)\big|_{y\geq 0} = 0 \\ \lim\limits_{y\to+\infty} \varpi(t,y)\big|_{t\geq 0} = 0 \\ \lim\limits_{y\to+\infty} \dfrac{\partial\varpi(t,y)}{\partial y}\Big|_{t\geq 0} = 0 \\ \varpi(t,0)\big|_{t\geq 0} = F(t) \end{cases} \tag{21}$$

where F is a given function which traduces plate intrinsic motion.

The solution process is applied to the particular case of opposite velocity-acceleration trends motion, traduced by:

$$F(t) \equiv A - Be^{-\beta t} \tag{22}$$

where A, B and β are positive real constants.

In this case, a classical analytical resolution protocol [26, 27] yielded the approximated solution:

$$\varpi(t,y) = A - Be^{-\beta t} - \frac{2}{\pi}\int_0^{+\infty}(A + \frac{1}{1+l\times\mu^2}\int_0^t B\beta e^{-\beta\theta} e^{\frac{k\mu^2}{1+k\mu^2}} d\theta\, d\mu \tag{23}$$

The BPES resolution protocol proposes a solution of the kind:

$$\varpi(t,y) = \frac{1}{2N_0}\sum_{k=1}^{N_0}\lambda_k \times B_{4k}(t\times r_k)\sum_{k=1}^{N_0}\tilde{\lambda}_k \times B_{4k}(y\times r_k) \tag{24}$$

where B_{4k} are the 4k-order Boubaker polynomials, r_k are B_{4k} minimal positive roots, N_0 is a prefixed integer and $\lambda_k, \tilde{\lambda}_k\big|_{k=1..N_0}$ are unknown pondering real coefficients.

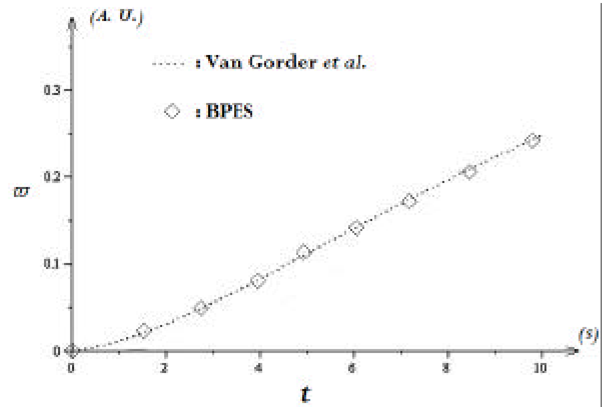

Fig. 2: Proposed solution to the problem of flat plate impulsive motion

This expression verifies the given conditions prior to resolution process. Partial plots of the solution are presented in Fig. 2, along with those proposed by Van Gorder *et al*. [29] for A=B=β=1.

Application to the problem of a beam free longitudinal vibrations inside a non-linear elastic medium: The main equation verified by a beam free longitudinal vibrations amplitude u inside a given non-linear elastic medium [30-32] is:

$$\begin{cases} \dfrac{\partial^2 u}{dt^2} + F(u) = q^2\dfrac{\partial^2 u}{dy^2} \\ q = \sqrt{\dfrac{E}{\rho}} \end{cases} \tag{25}$$

Where ρ is the density of beam, E is Young's modulus, q is the velocity of wave propagation and F is the restoring force per unit mass acting on the beam from the surrounding medium.

The targeted solution is subjected to the boundary conditions:

$$\begin{cases} u(z,0) = f(z) \\ \dfrac{\partial u(z,t)}{\partial t}\Big|_{t=0} = g(z) \end{cases} \tag{26}$$

where f and g are given functions and:

$$\begin{cases} u(0,t) = 0 \\ \dfrac{\partial u(z,t)}{\partial z}\Big|_{z=l} = 0 \end{cases} \tag{27}$$

where l is the length of the beam.

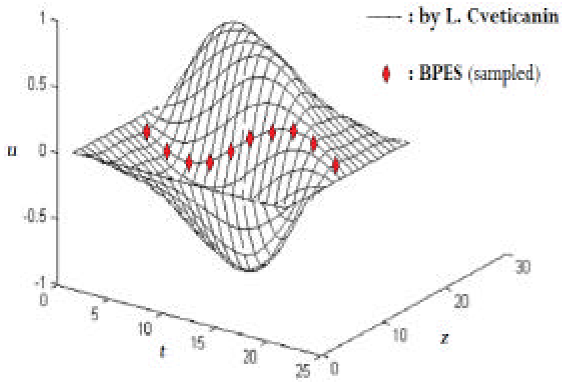

Fig. 3: Proposed solution to the problem of beam free longitudinal vibrations inside a non-linear elastic medium

The solution process is applied to the particular case of a symmetric nonlinear force:

$$F(u) \equiv b_1 u + b_3 u^3 + b_5 u^5 + b_7 u^7 + ... \tag{28}$$

with, as an analytical solution [31] is:

$$u(z,t) = \sum_{k=1}^{+\infty} d_k \sin\omega_k t \times \sin(kz) \tag{29}$$

where:

$$d_k = \frac{2}{\pi\omega_k}\int_0^{\pi} g(z) \times \sin(kz) \tag{30}$$

The BPES-related resolution protocol proposes also the solution:

$$u(z,t) = \frac{1}{2N_0}\sum_{k=1}^{N_0}\lambda_k \times B_{4k}(z \times r_k)\sum_{k=1}^{N_0}\tilde{\lambda}_k \times B_{4k}(z \times r_k) \tag{31}$$

which verifies the given conditions prior to resolution process.

Plots of the solution are presented in Fig. 3, along with those proposed by L. Cveticanin [30-32].

CONCLUSION

The Boubaker polynomials and the BPES have been studied. This is of interest not only because of their applications in nonlinear applied physics fields, but also because the used method can be applied to solve problems in chemistry, biology, mechanics and medicine. We have presented the features of the related Boubaker Polynomials Expansion Scheme (BPES) and discussed some of its applications.

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