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A Modified Economic-Statistical Design of T^2 Control Chart with Variable Sampling Intervals, Control Limits and Warning Lines

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Abstract: The usual sampling method to monitor a process when more than a quality characteristic is being observed, is Hotelling's T^2 control chart with Fixed Ratio Sampling (FRS) scheme. Recent researchers have shown that the T^2 chart with Variable Sampling Intervals and Control Limits (VSIC) detects a small shift in the process mean faster than the classical one and it has been shown that the VSIC scheme is more economical than FRS scheme. Furthermore, statistical (and economical) comparison between VSIC scheme and Variable Sampling Intervals (VSI) scheme has been done. Faraz and Saniga [1], studied economic-statistical design of VSI-T^2 chart. They assumed the process parameters (μ , Σ) are known and employed the cost model proposed by Lorenzen and Vance [2]. This paper represents an economic-statistical design of a VSIC with variable warning lines. We suppose that μ and Σ are unknown, which is the same in practice and we use the cost model proposed by Costa and Rahim [3] and find minimum cost using a Genetic Algorithm (GA) approach. Ultimately, we do meaningful and unbiased comparisons between VSIC (with one and two warning lines), VSI and FRS schemes.

Key words: Variable Sampling Intervals and Control Limits (VSIC) with variable warning lines . Multivariate Control Chart . Economic-Statistical Design . Markov Chain (MC) . Genetic Algorithm (GA) . Adjusted Average Time to Signal (AATS)

INTRODUCTION

The source of changes in a production process contains random and assignable causes. Statistical Process Control (SPC) are used to monitor processes to detect assignable causes such as worker errors, machine wear, or changes in the quality of raw materials.

To design a control chart, we consider three following parameters:

1. The sample size (n)
2. Sampling interval (h)
3. The width of control limit (k);

which usually are specified according to statistical or/and economic criteria. The statistical performance of a control chart, generally refers to the time required by it to detect an out-of-control condition and the economic performance refers to the process control related cost due to the chart.

In a statistically designed control chart the design parameters are chosen such that the chart meets some statistical performance requirements, while the minimization of the net sum of all costs involved yields an economic design. For an economic-statistical design the parameters are chosen to minimize the costs subject to some constraints on the statistical performance. Most of the processes are characterized by several variables. These random variables are usually correlated and assumed following a multivariate Normal distribution (for more information of multivariate control chart; see Jackson [4] and Lowry and Montgomery [5]). The most famous multivariate control chart to monitor the processes mean is the Hotelling T^2 control chart [6]. Since it satisfies three important properties, multivariate techniques should possess [4] and also has the additional advantage of simplicity among the other available multivariate control charts.

The traditional sampling method in the Hotelling T^2 control chart is the Fixed Ratio Sampling (FRS) scheme, in which all of the design parameters are fixed to monitor a process. The FRS T^2 control scheme has good performance in detecting large shifts in the

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process mean, but it's performance to detect small or even moderate shifts as quickly as possible may be poor. The FRS control charts are static; that is, all the three design parameters are constant during the time of monitoring a process.

Reynolds *et al*. [7] were the first to consider an adaptive strategy for the sampling interval based on the data trends in monitoring a univariate process, that is, Variable Sampling Intervals (VSI) $\bar{X}$ chart. To be adaptive, for a control chart, means that; at least one of the design parameters can change its value for the current sample depending on the location of the past sample point on the chart. Aparisi extended the idea of adaptive strategy in univariate to multivariate case. He considered, statistical design of T^2 control charts with Variable Sample Sizes (VSS), Variable Sampling Intervals (VSI) Variable Sample Sizes and Sampling Intervals (VSSI) [8-10]. Ultimately they indicated that among the FRS, VSI, VSS and VSSI T^2 charts, the VSI-T^2 chart is the quickest to detect large shifts in the process mean vector, meanwhile the VSSI-T^2 chart is the quickest one to detect small shifts. Nevertheless, both of the charts have almost similar statistical performance to detect moderate shifts.

It is remarkable that, they assumed μ and Σ are known and for simplicity the process starts from an out-of-control state (d>0). They also considered ATS (average time to signal) criterion for the statistical efficiency of control chart. Hence they obtained designs with extremely long sampling intervals which kept them unpractical. Faraz and Moghadam [11], Faraz *et al*. [12] solved this problem. They statistically designed the VSS-T^2 and VSI-T^2 control charts based on the Adjusted Average Time to Signal (AATS) measure, respectively. They assumed the process starts from an in-control state (d = 0) and the amount of time that the process remains in control has exponential distribution. Faraz and Parsian [13] offered statistical designed VSS and/or VSI T^2 control chart with Double Warning Line (DWL). They have shown that the DWL scheme detects process mean shifts more quickly than VSS, VSI and VSSI T^2 charts. Finally, Chen and Hsieh considered new chart termed Variable Sample Size and Control Limit (VSSC) and showed that for detecting a small shift in the process mean it is quicker than the VSS, VSI, VSSI and FRS T^2 charts.

Nevertheless; none of the mentioned papers considered the economic aspect of the control chart. Economic Design (ED) is an alternative to statistical design of control charts. Three kinds of costs can be considered in the ED of control charts:

1. The cost of sampling and testing;
2. The cost of investigating an out of control signal and repairing any assignable cause found;
3. The cost of producing defective products;

which all of them depend on the control chart parameters. In the ED of control charts, cost is a critical criterion for measuring control effectiveness and chart's parameters are determined by minimizing the overall costs associated with maintaining current control of a process.

For the first time Weiler [14] proposed ED of control chart based on the total number of inspected items required to detect a shift in the process mean. In the same year Girschick and Rubin [15] exhibited an economic design based upon the Expected Cost per Unit Time (ECT). Dunkan [16] then proposed a known model for the ED of Shewart $\bar{X}$ chart, which influenced all future related researches. In this model the process is continuous i.e. the process continues during the search for detecting assignable cause and it is assumed that the length of time that the process remains in control is described by an exponential distribution with constant hazard rate.

Following Dunkan's work, due to the importance of the subject a lot of research have been done on the ED of $\bar{X}$ chart. Thirty years later, Loranzen and Vance [2] extended an economic model which can be applied for all kinds of Shewart control charts. Despite of the usefulness of these models for FRS scheme, their weakness is that their application to Variable Ratio Sampling (VRS) designs is not simple.

Hence, Costa and Rahim [3] developed an economic model based on the Markov chain approach which is suitable to study the economic design of the VRS schemes [17]. In this model, during the search for an assignable cause, the process is stopped. Montgometry [18] and Ho and Case [19] provided a literature review on the ED of control charts. For the first time, Montgomery and Klatt [20] extended the ED of control chart for the FRS-T^2 control chart. They supposed that only one assignable cause of variation exists and the time between occurrences has exponential distribution with constant hazard rate.

Recently, Chou *et al*. [21] applied the cost model proposed by Montgomery and Klatt [20] to ED the VSI-T^2 control charts. In the same year, Chen [22] presented the ED of VSI-T^2 control charts with Dunkan's cost model [16]. Then he [23] studied ED of T^2 control chart with the VSSI sampling scheme and applied Costa and Rahim [3] cost model for that. Faraz *et al*. [17] had a survey on the economic properties of the T^2 control chart with adaptive sample size. It is worth to mention that the study in this field still continues.

Unfortunately, economic designs of the control charts do not mention the statistical performance. Woodal [24] showed the EDs have relatively larger probability of type ***I*** decision in process control. Saniga [25] removed this problem by extending an approach named Economic Statistical Design (ESD) of control charts. In this procedure statistical constraints are placed on the proposal cost model of ED. Using a large experiment, he found that the ESDs have slightly higher costs than ED and their statistical properties are as good as statistically designed control charts. In addition ESDs have the counter intuitive property that one can, at time, reduce cost by tightening statistical constraints [1].

Seif *et al*. [26] survived ESD of T^2 control charts using Variable Sample Sizes and control limits (VSSC). They did meaningful and unbiased comparisons between VSSC and VSS T^2 charts. Torabian *et al*. [27] economically designed the Variable Sampling Intervals and Control Limits (VSIC) and they compared VSIC and VSI T^2 charts with respect to the ECT. They showed that the expected loss of the VSIC scheme with two warning lines is little smaller than the VSI scheme, but it is not economically better than the VSIC with one warning line. They did not consider statistical discretion.

Ultimately, Faraz and Saniga [1] investigated the ESD for the T^2 control chart with VSI sampling scheme on the cost model proposed by Lorenzen and Vance [2]. In their paper, they supposed process parameters (μ_0, S) are known. In addition, they concluded that one can meet the same statistical constraints as statistical designs with ESD VSI designs but with large reductions in cost and the VSI designs are economically dominant when compared to FRS designs [1].

In this paper we examine and modify economic statistical design of T^2 control chart with variable sampling intervals, control limits and warning lines, a problem up to now not addressed. We consider process parameters are unknown which is the usual case in practice. The cost model we apply here is the general model of Costa and Rahim [3], which is suitable to investigate economical design of the VRS scheme and we optimize this model by using a Genetic Algorithm (GA) approach. This paper is organized as follows. In section 2, the VSIC T^2 control chart and Makov chain approach are briefly reviewed. In section 3, the cost model proposed by Costa and Rahim [3] has been described for our situation and the solution procedure of the proposed cost model using GA is discussed in section 4. Section 5 contains a description of the proposed procedure to find economical statistical designs so that a comparison of the FRS and VSIC schemes could be meaningful and unbiased. Section 6 provides numerical comparisons between VSICC (one and two scales), VSI and FRS schemes. Finally, concluding remarks make up the last section.

VSIC T^2 CONTROL SCHEME AND MARKOV CHAIN APPROACH

In order to control a process with p>1 correlated characteristics using the T^2 control charts, we make the usual assumption that the join probability distribution of the quality characteristics is a p-variate normal distribution with in-control mean vector $\mu' = (\mu_{01},\ldots,\mu_{0p})$ and variance-covariance matrix Σ . Then the subgroup (each of size) statistics

$$T_i^2 = n(\overline{X}_i - \mu_0)'\Sigma^{-1}(\overline{X}_i - \mu_0)$$

are plotted in sequential order to form the T^2 control chart and the chart signals as soon as $T_i^2 \geq k$ [1].

Clearly when the process parameters μ_0 and Σ are known, the control limit k is equal to the upper ath quantile of central chi-square distribution with p degrees of freedom, i.e. $k = \chi_\alpha^2(p)$.

It is an usual fact in practice that μ_0 and Σ are unknown, thus they first are estimated through m initial samples, each of size n, during a time that the process is in-control. The unbiased estimators of the unknown parameters μ_0 and Σ , respectively, consist of:

$$\begin{cases} \overline{\overline{X}} = \dfrac{\sum_{j=1}^{m} \overline{X}_j}{m} & s.t.: (\overline{X}_j = \dfrac{\sum_{k=1}^{n} X_{jk}}{n}: \quad j = 1,\ldots,m) \\ \overline{S} = \dfrac{\sum_{j=1}^{m} S_j}{m} & s.t. (S_j = \dfrac{\sum_{k=1}^{n} (X_{jk} - \overline{X}_j)(X_{jk} - \overline{X}_j)'}{n-1}: j = 1,\ldots,m) \end{cases} \tag{1}$$

Since for n>1, $\dfrac{m(n-1)-p+1}{p(m+1)(n-1)}$ T^2 is distributed as F distribution with p and m(n-1)-p+1 degrees of freedom, then for next subgroups each of size n>1 the $T_i^2 = n(\overline{X}_i - \overline{\overline{X}})'\overline{S}^{-1}(\overline{X}_i - \overline{\overline{X}})$ values are compared with:

$$k = \frac{p(m+1)(n-1)}{m(n-1)-p+1} F_{\alpha}(p, m(n-1)-p+1) \qquad (2)$$

where $F_{\alpha}(p, m(n-1)-p+1)$ is the upper ath quantile of central F-distribution with p and m(n-1)-p+1 degrees of freedom.

Furthermore if n = 1, then the control limit is equal to:

$$k = \frac{p(m^2-1)}{m(m-p)} F_{\alpha}(p, m(m-p)) \qquad (3)$$

For simplicity we apply the same notation used by Chen [22] and [28], i.e.

$$C(m,n,p) = \begin{cases} \frac{p(m+1)(n-1)}{m(n-1)-p+1} & : n > 1 \\ \frac{p(m^2-1)}{m(m-p)} & : n = 1 \end{cases} \qquad (4)$$

$$v = \begin{cases} m(n-1)-p+1 & : n > 1 \\ m(m-P) & : n = 1 \end{cases} \qquad (5)$$

In general equations (2) and (3) can be written as:

$$k = C(m,n,p) F_{\alpha}(p,v) \qquad (6)$$

Clearly if the number of initial subgroups, m, is large enough then the exact control limit tends to chi-square control limit [5]. It is always recommended m>20 and often m>50.

As we know when a FRS T^2 control chart is used to monitor a multivariate process, a sample of size n is drawn every h_0 hours and the value of the T^2 statistic which is said sample point, is plotted on a control chart and is compared with control limit:

$$k_0 = C(m, n_0, p)\, F_{\alpha}(p,v)$$

The VSIC-T^2 control chart is a modification of the FRS-T^2 control chart. For this purpose, let h_1 and h_2 be the maximum and minimum sampling intervals, also k_1 and k_2 be the maximum and minimum control limits, respectively, such that

$$0<h_2<h_0<h_1 \text{ and } 0<k_2<k_0<k_1$$

The decision to switch between h_1 and h_2 sampling intervals and k_1 and k_2 control limits, for the current sample point (i.e. T_i^2) depends on the position of the prior sample point (i.e. T_{i-1}^2) on the control chart. If T_{i-1}^2 falls in the safe (or relaxing) region, the maximum sampling interval h_1 and maximum control limit k_1 will be used for the current sample point i. On the other hand, if T_{i-1}^2 falls in the warning (or tightening) region, the minimum sampling interval h_2 and minimum control limit k_2 will be used for the current sample point i. Finally, if T_{i-1}^2 falls in the action region, then the process is considered, out of control. Here, the safe, warning and action regions are defined by the warning limit w_j and the control limit k_j and are $[o,w_j)$, $[w_j, k_j)$ and $[k_j, \infty)$ intervals, respectively, where j = 1 if the prior sample point comes from the long sampling interval h_1 and j = 2 if the prior sample point comes from the short sampling interval h_2. The following function summarizes the control scheme of the VSIC-T^2 control chart.

$$(h_{(i)}, k_{(i)}) = \begin{cases} (h_1, k_1) & \text{if} \quad 0 \le T_{i-1}^2 < w_{(i-1)} \\ (h_2, k_2) & \text{if} \quad w_{(i-1)} \le T_{i-1}^2 < k_{(i-1)} \end{cases} \qquad (7)$$

In the literature, the most recently used measure to compare the efficiencies of control schemes with different sampling strategies is the adjusted average time to signal which is shown by AATS. This statistical criterion is defined as the average time from a process mean shift until a signal in chart occurs; and is equal to

$$AATS = ATC - (Expected\ lengh\ of\ in - control\ time) \qquad (8)$$

where ATC (the average time of the cycle) is the average time from the start of the production until the first signal; after the process mean shift. We suppose the assignable cause occurs according to an exponential distribution with parameter λ. Thus the expected length of in-control period is 1/λ and the following equation is obtained:

$$AATS = ATC - \frac{1}{\lambda} \qquad (9)$$

By considering the memory less property of the exponential distribution, we can compute ATC (and also AATS) using the Markov Chain (M.C.) approach. Here, the applied M.C. approach, is similar to that of Faraz and Saniga [1].

The fundamental concepts of the Markov chain approach applied in following, can be seen in Cinlar [29]. For our situation, each sampling stage can be considered as one of the following five transient states that depends on the states of the process (in or out-of-control) and the position of T_i^2 in the control chart (safe, warning or action region). For j = 1,2:

State 1: $0 \le T_i^2 \le w_j$ and the process is in control;

Table 1: The state of the M.C. Position of T_i^2 for i-th sampling

	In-control	Out-of-control
Safe region	State 1	State 4
Warning region	State 2	State 5
Action region	State 3	State 6

State 2: $w_j \le T_i^2 < k_j$ and the process is in control;

State 3: $T_i^2 \ge k_j$ and the process is in control (false alarm);

State 4: $0 \le T_i^2 < w_j$ and the process is out-of-control;

State 5: $w_j \le T_i^2 < k_j$ and the process is out-of-control;

The absorbing state (state 6) is reached when $T_i^2 \ge k$ and the process is out-of control (true alarm). Here, we can see the states of M.C.approach in Table 1.

When sample point T_i^2 falls into the action region, the control chart produce a signal. Hence when state 3 is reached, the signal is a false alarm and when absorbing state 6 is reached, the signal is true alarm. The transition probability matrix is given by

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & p_{46} \\ 0 & 0 & 0 & p_{54} & p_{55} & p_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (10)$$

where p_{ij} denotes the transition probability that i is the prior state and j is the current state. By considering

$$a_i = F(\frac{w_i}{C(m.n.p)}, p, v, \eta = 0);$$

$$b_i = F(\frac{k_i}{C(m.n.p)}, p, v, \eta = 0);$$

$$c_i = F(\frac{w_i}{C(m.n.p)}, p, v, \eta = nd^2);$$

$$d_i = F(\frac{k_i}{C(m.n.p)} p, v, \eta = nd^2); (i = 1,2)$$

Then p_{ij}'s are

$$P_{11} = a_1 q_1$$
$$P_{12} = (b_1 - a_1) q_1$$
$$P_{13} = (1 - b_1) q_1$$
$$P_{14} = c_1 (1 - q_1)$$
$$P_{15} = (d_1 - c_1)(1 - q_1)$$
$$P_{16} = (1 - d_1)(1 - q_1)$$
$$P_{21} = P_{31} = a_2 q_2$$
$$P_{22} = P_{32} = (b_2 - a_2) q_2$$
$$P_{23} = P_{33} = (1 - b_2) q_2$$
$$P_{24} = P_{34} = c_2 (1 - q_2)$$
$$P_{25} = P_{35} = (d_2 - c_2)(1 - q_2)$$
$$P_{26} = P_{36} = (1 - d_2)(1 - q_2)$$
$$P_{44} = c_1$$
$$P_{45} = d_1 - c_1$$
$$P_{46} = 1 - d_1$$
$$P_{54} = c_2$$
$$P_{55} = d_2 - c_2$$
$$P_{56} = 1 - d_2$$

Here, the notation, F(x,p,v,η) represents the non-central F distribution function with p and v degrees of freedom and non-centrality parameter v= nd^2 where

$$d = \sqrt{(\mu_1 - \mu_0)' \Sigma^{-1} (\mu_1 - \mu_0)}$$

is Mahalanobis distance (the density function of non-central F distribution can be seen in Anderson [30]).

Since the expected number of trials needed in each state to reach the absorbing state 6 is equal to **b'(I-Q)$^{-1}$** where **I** is the identity matrix of order 5, Q is the 5×5 matrix obtained from transition probability matrix **P** by deleting the elements corresponding to the absorbing state 6 and b'= (p_1, p_2, p_3, p_4, p_5) is the vector of initial probabilities with $\sum_{i=1}^{5} p_i = 1$ [29].

Hence ATC criterion is obtained as follows:

$$ATC = b'(I - Q)^{-1} h \quad (11)$$

where h′ = (h_1, h_2, h_2, h_1, h_2) is the vector of sampling time intervals. The third element in h′ vector is replaced by h_2 in order to provide an additional protection to prevent problems that arise during start up [22].

In this paper, as almost recent papers, we suppose that the process starts up at state 2. In order words, we set the vector b′ = (0,1,0.0.0) to provide an extra protection and to prevent encountered problems during start-up.

THE COST MODEL

This section consists of two parts; model assumption and the loss function.

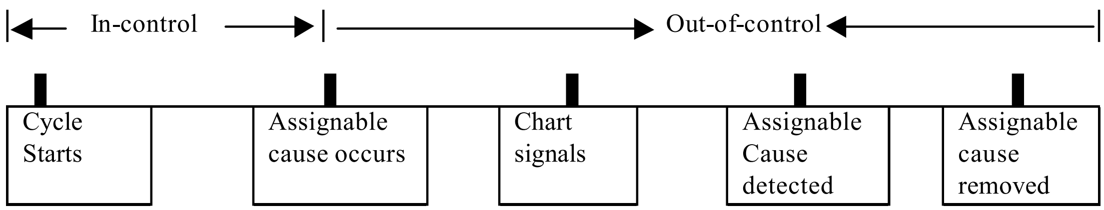

Fig. 1: A quality cycle

Model assumptions: The Costa and Rahim [3] model is an extension of Duncan's [16] for economic design of control chart based on Markov Chain approach, which is suitable to study the economic design of the VRS schemes. To construct our process control model by a VSIC-T^2 control chart, we consider the following usual assumptions:

1. The P quality characteristics follow a multivariate normal distribution with mean vector μ and covariance matrix Σ .
2. The process is characterized by an in-control state $\mu = \mu_0$.
3. Only one single assignable cause produces "step changes" in the process mean from $\mu = \mu_0$ to a known $\mu = \mu_1$. (This results in a known value of the Mahalanobis distance (d)).
4. "Drifting processes" are not a subject of this research. i.e., assignable causes that affect process variability are not addressed; hence it is supposed that the covariance matrix Σ is constant over time.
5. The assignable cause is assumed to occur according to a Poisson process with intensity λ occurrences per hour. That is assuming that the process starts in the in-control state, the time interval that the process remains in-control is an exponential random variable with mean $1/\lambda$.
6. The process is not self correcting. That is, once a transition to an out-of-control state has occurred, the process can be returned to the in-control condition only by management intervention upon appropriate corrective action.
7. The quality cycle starts with the in-control state and continues until the process is repaired after an out-of-control signal. It is assumed that quality cycle follows a renewal reward process.

The loss function: Consider the quality cycle as the time between the starts of successive in-control period. An entire quality cycle is represented in Fig. 1.

The process cycle is divided into four following time intervals:

a) The in-control period (i.e. the time until an assignable cause occurs).
b) The time until the chart gives an out-of control signal.
c) The time to take a sample and interpret the results.
d) The time to discover the assignable cause and repair the process.

The expected time interval that the process remain in-control, is equal to:

$$In-control period = \frac{1}{\lambda} + T_0 \times ANF \quad (12)$$

where T_0 is the average search time for a false alarm and ANF is the average number of false alarms in each quality cycle and is calculated as follows:

$$ANF = \mathbf{b'(I\text{-}Q)^{-1}(0,0,1,0,0)} \quad (13)$$

Furthermore, let T_1 be the average time to find the assignable cause and repair the process, in this case the average time of, out-of-control, is given by

$$out-of-control period = AATS + T_1 \quad (14)$$

Therefore the expected time of a quality cycle is sum of (12) and (14) values, i.e.:

$$E(T) = \frac{1}{\lambda} + T_0 \times ANF + AATS + T_1 \quad (15)$$

$$= ATC + T_0 \times ANF + T_1 \quad \text{(by (6))}$$

On the other hand, if we define V_0 and V_1 as the average profit per hour earned when the process is operating in-control and out-of-control states, respectively, C_0 as the average consequence cost of a false alarm, C_1 as the average cost to find the assignable cause and repair the process and s as average cost for each inspected item; then the expected cost per quality cycle is determined as follows :

$$E(C) = V_0(1/\lambda) + V_1 \times AATS - C_0 \times ANF - C_1 - s \times ANI \quad (16)$$

The ANI and ANS are average number of inspected items and samples taken during the quality cycle and are calculated as follows:

$$ANS = b'(I - Q)^{-1}1 \quad (17)$$

$$ANI = n \times ANS \quad (18)$$

Finally, based on the renewal reward process assumption, the expected loss per hour E(L) is given by

$$E(L) = V_0 - \frac{E(C)}{E(T)}$$

$$= V_0 - \frac{V_0(\frac{1}{\lambda}) + V_1 \times AATS - C_0 \times ANF - C_1 - s \times ANI}{ATC + T_0 \times ANF + T_1} \quad (19)$$

SOLUTION TO THE COST MODEL

In the ED of VSIC-T^2 control chart, it is assumed that the five process parameters (p, λ, d, T_0, T_1) and the five cost parameters (v_0, v_1, c_0, c_1, s) are previously estimated. Then, the solution procedure finds the seven chart parameters (k_1, k_2, w_1, w_2, n, h_1, h_2) which minimize E(L) in (19). Among these seven chart parameters, the sample size n is a discrete variable and the other six variables are continuous where $0 \le w_2 \le w_1 < k_2 < k_1$. Thus, the general optimization problem is defined as follows

$$\begin{aligned}
&\min E(L)\\
&\text{s.t.:}\\
&0 < k_2 < k_1\\
&0 \le w_1 < k_1\\
&0 \le w_2 < k_1 \qquad (20)\\
&w_2 \le w_1\\
&0.01 \le h_2 \le h_1 \le 8 (\textit{to keep the chart practical})\\
&n \in Z^+\\
&ANF \le 0.5
\end{aligned}$$

To provide the best protection against false alarms and/or to detect process shifts as quickly as possible, the statistical constraints, $ANF \le ANF_0$ and/or $AATS \le AATS_1$, respectively; can be added to form an ESD. It is obvious that the minimization problem (20) has both discrete and continuous decision variables and a discontinuous and non-convex solution space. Just like the recent extensive papers we find the optimal values of model parameters using the GA approach [1, 21-23, 32-34].

GAs are search algorithms that were developed based on an analogy with natural selection and population genetics in biological system [31]. The operations of GAs possess four steps:

1. Randomly generate an initial solution population of candidate solutions (k_1, k_2, w_1, w_2, n, h_1, h_2), each one is represented as a string of bits.
2. Assign each bit string a value according to a fitness function (i.e., the objective function that minimizes the E(L)) and select strings from old population randomly but biased by their fitness.
3. Recombine these strings by using the cross over mutation operators.
4. Produce a new generation of strings that are more fit than the previous one.

The termination condition is achieved when the number of generations is large enough or a satisfied fitness value is obtained. The following settings of parameters for the package manipulation have been used:

Population size (N_{pop}) was set up to 50;
Crossover probability (r_c) was set up to 0.5;
Mutation rate (r_m) was set up to 0.25;

And the number of generation was set at least 100,000 times. For the details of our solution method the reader is referred to Faraz *et al*. [12].

COMPARISON APPROACH

As Seif *et al*. [26] and Faraz and Saniga [1] mentioned, two different schemes FRS and VSIC (two scales) should have the same comparing measure when the process is in-control. To compare the two different schemes in a fair way, the VSIC-T^2 chart is designed such that it has the same expected cost per hour as the FRS_T^2 chart, when the process is in-control state. By considering that the two schemes have the same in-control time, then the two charts are fairly comparable if and only if they have the same in-control cycle cost. Then, the two charts should have the same ratio sampling (sampled item and sampling frequencies) and the same Type I error rate as long as the process is in-control [26, 35, 36]. By letting $1'$= (1,1,1,0,0) in (17) and (18), the ANS and ANI for the VSI_T2 control chart are obtained as follows:

$$in-control-ANS_{VSIC} = b'(I-Q)^{-1}(1,1,1,0,0) \quad (21)$$

$$\begin{aligned} in-control-ANI_{VSIC} \\ = n \times (In-control-ANS_{VSIC}) \end{aligned} \quad (22)$$

On the other hand, considering the set of parameters (k_0, n_0, h_0) for the FRS_T^2 control chart, we can easily show that the in-control ANS and ANI for the FRS scheme are obtained by [12].

$$In-control-ANS_{FRS}=\frac{1}{1-e^{-\lambda h_0}} \quad (23)$$

$$In-control-ANI_{FRS}=\frac{n_0}{1-e^{-\lambda h_0}} \quad (24)$$

It can be observed easily that the equality of In-control-ANS for two different schemes will conclude the equality of In-control-ANI. Now, in the VSSC design vector (k_1, k_2, w_1, w_2, n, h_1, h_2), the value of w is obtained such that the both VSIC and FRS schemes have the same in-control ANS. Hence by equating expressions (21) and (23) an considering C = C(m,n,p), the parameter w_1 is obtained as follows:

$$w_1=CF^{-1}\left(\frac{F(\frac{w_2}{C},p,v,o)e^{-\lambda h_2}(e^{-\lambda h_1}-e^{\lambda h_0})+e^{-\lambda h_2}-e^{-\lambda h_0}}{e^{-\lambda h_1}(e^{-\lambda h_2}-e^{\lambda h_0})}\right) \quad (25)$$

On the other hand the VSIC scheme should have the average Type I error rate equals to α_0 and the average sampling interval equals to h_0 during in-control period. Now, assume that the probability of having the minimum sampling plan while the process is in-control is p_0. So, the maximum sampling plan occurs with the probability (q_0 = 1-p_0) as long as the process is in-control. Therefore, we should have

$$\begin{cases}\alpha_1 p_0+\alpha_2 q_0=\alpha_0\\ h_1 p_0+h_2 q_0=h_0\end{cases} \quad (26)$$

where α_i = 1-F(k_i, p, 0); i = 0,1,2. Hence, the expression for the calculation of k_2 is obtained as follows:

$$k_2=CF^{-1}\left(\frac{F(\frac{k_0}{C},p,v,0)-p_0F(\frac{k_1}{C},p,v,0)}{1-p_0},p,v,0\right) \quad (27)$$

where

$$p_0=\frac{h_0-h_2}{h_1-h_2} \quad (28)$$

Consequently, for a given process and cost parameters, the optimal design of the FRS-T^2 control scheme (k_0, n_0, h_0) is first defined. Then for a given (k_1, h_1, h_2, w_2) parameter set, the parameters w_1 and k_2 take value from equations (25) and (27), respectively. Then we proceed to find the four chart parameters (k_1, h_1, h_2, w_2) that minimize (20). Then procedure insures that the comparison of the favored different schemes is meaningful and unbiased because the two FRS and VSIC schemes have the same cost while the process is in-control. The proposed scheme is illustrated through an industrial application.

NUMERICAL COMPARISON

The T^2 control chart with the VSIC (one and two scales), VSI and FRS are compared with respect to the lost in this section. We use the thirteen process and cost parameters in Table 2, which adapted from the studies of the univariate control chart reported by Costa and Rahim [3]. These values provide a general variation in parameter values. Table 3 and 4, show the optimal design parameters and the loss for the VSI, FRS and VSIC (two scales) schemes for the two cases (p = 2, m = 25) and (p = 4, m = 50), respectively. Given the first parameter set, for instance, the optimal design parameters (k = 14.15, w = 4.00, h_1 = 5.49, h_2 = 0.01, n = 12) with minimal loss 38.47 are found when n and k are fixed. Also for fixed ratio sampling minimum loss 43.56 is reached. Likewise, a minimal value of 37.57 is obtained when the optimal design parameters n is fixed (n = 12) but k,w and h vary between (k_1 = 15.92, k_2 = 11.98, w_1 = 4.62, w_2 = 2.54) and (h_1 = 5.01, h_2 = 0.01), respectively. The results are opposed to what we expected. The results show that the expected loss of the VSIC (two scales) scheme, is smaller than FRS and, with not large differences, is almost smaller than the VSI scheme. It is remarkable that, the VSIC scheme with two warning lines imposes some difficulties in application. Hence, for simplicity , we also consider the VSIC scheme with one warning line, i.e. w_1 = w_2, which enable us to apply one measurement scale. Table 5 and 6 give the optimal design parameters and resulting expected loss for the VSI, FRS and VSIC (one scale) schemes for the two cases (p = 2, m = 25) and

Table 2: The 13 process and cost parameters

NO	S	C_0	C_1	V_0	V_1	T_0	T_1	λ	d
1	5	500	500	500	50	5	1	0.01	1
2	10	500	500	500	50	5	1	0.01	1
3	5	250	500	500	50	5	1	0.01	1
4	5	500	50	500	50	5	1	0.01	1
5	5	500	500	250	50	5	1	0.01	1
6	5	500	500	500	100	5	1	0.01	1
7	5	500	500	500	50	5	1	0.01	1
8	5	500	500	500	50	2.5	1	0.01	1
9	5	500	500	500	50	5	1	0.01	1
10	5	500	500	500	50	5	10	0.05	1
11	5	500	500	500	50	5	1	0.01	1.5
12	5	500	500	500	50	5	1	0.01	0.5
13	5	500	50	500	50	5	1	0.01	2.0

Table 3:The optimal parameters of the economic statistical design of the FRS , VSI and VSIC Schemes with two warning limits for the case p = 2 and m = 25

	VSI scheme							FRS	VSIC scheme								
NO	k	w	h_1	h_2	n	ANF	E(L)	E(L)	k_1	k_2	w_1	w_2	h_1	h_2	n	ANF	E(L)
1	14.15	4.00	5.49	0.01	12	0.03	38.47	43.56	15.92	11.98	4.62	2.54	5.01	0.01	19	0.01	37.57
2	12.23	3.51	7.20	0.01	10	0.06	49.53	54.83	13.85	10.68	4.26	2.61	6.88	0.01	16	0.05	48.47
3	13.74	6.67	5.28	0.01	11	0.04	38.29	43.34	16.11	11.63	4.43	2.30	4.72	0.01	18	0.03	36.96
4	13.91	3.65	5.26	0.01	11	0.04	34.07	39.28	16.31	11.54	4.44	2.28	4.71	0.01	18	0.04	33.04
5	12.46	3.49	7.74	0.01	10	0.05	26.12	28.65	14.27	10.80	4.28	2.59	7.38	0.01	16	0.03	27.57
6	14.12	4.01	5.84	0.01	12	0.03	36.84	41.54	15.98	12.02	4.62	2.58	5.4	0.01	18	0.03	35.95
7	14.18	3.99	5.19	0.01	12	0.03	40.00	45.44	16.07	11.97	4.65	2.54	4.81	0.01	18	0.02	39.15
8	12.74	3.42	5.04	0.01	10	0.07	37.86	42.38	15.02	11.09	4.33	2.39	4.77	0.01	17	0.04	36.93
9	14.01	4.02	5.75	0.01	12	0.03	75.06	79.39	15.87	11.81	4.60	2.61	5.31	0.01	18	0.01	74.39
10	13.04	3.84	2.57	0.01	11	0.02	107.90	114.57	14.14	11.52	4.50	3.07	2.48	0.01	17	0.01	107.29
11	16.67	4.27	3.86	0.01	6	0.02	29.47	33.86	19.33	14.02	5.33	2.87	3.69	0.01	9	0.01	29.15
12	10.36	3.31	9.63	0.01	35	0.10	64.53	69.05	11.26	9.11	3.79	2.62	9.13	0.01	50	0.07	64.34
13	18.91	3.49	2.80	0.01	3	0.02	25.10	28.69	22.83	16.05	6.01	3.30	3.03	0.01	6	0.02	24.91

Table 4:The optimal parameters of the economic statistical design of the FRS, VSI and VSI& Schemes with two warning limits for the case p = 4 and m = 50

	VSI scheme							FRS	VSIC scheme								
NO	k	w	h_1	h_2	n	ANF	E(L)	E(L)	k_1	k_2	w_1	w_2	h_1	h_2	n	ANF	E(L)
1	18.00	6.78	5.95	0.01	14	0.03	41.21	46.00	20.17	16.02	7.55	5.03	5.45	0.01	22	0.02	40.63
2	16.20	6.65	8.19	0.01	13	0.05	53.57	58.21	17.37	14.63	7.40	5.54	7.88	0.01	19	0.03	53.12
3	17.57	6.42	5.76	0.01	13	0.04	41.11	45.80	19.91	15.88	7.55	5.06	5.52	0.01	22	0.03	40.37
4	18.01	6.78	5.92	0.01	14	0.03	36.92	41.47	20.21	16.03	7.54	5.00	5.45	0.01	22	0.02	36.22
5	16.44	6.62	8.81	0.01	13	0.05	28.01	30.23	17.93	15.01	7.42	5.51	8.47	0.01	19	0.03	27.82
6	17.96	6.79	6.33	0.01	14	0.03	39.43	43.85	20.02	15.89	7.55	5.04	5.87	0.01	22	0.02	38.64
7	18.03	6.77	5.63	0.01	14	0.04	42.89	48.03	20.08	16.05	7.56	4.98	5.16	0.01	22	0.03	42.11
8	17.03	6.91	5.92	0.01	14	0.05	40.76	44.81	18.82	15.05	7.50	5.23	5.52	0.01	20	0.03	40.06
9	17.59	6.42	6.04	0.01	13	0.04	77.46	81.55	19.86	15.93	7.52	5.08	5.76	0.01	21	0.02	76.82
10	17.01	7.03	2.90	0.01	14	0.02	113.98	119.36	17.80	15.44	7.34	5.77	2.73	0.01	21	0.02	113.40
11	20.59	7.19	4.17	0.01	7	0.02	31.21	35.41	23.11	18.04	8.40	5.46	4.05	0.01	11	0.01	30.80
12	13.93	6.07	10.74	0.01	43	0.11	70.33	74.61	14.83	13.01	6.53	5.25	10.29	0.01	50	0.09	69.92
13	22.67	7.09	3.16	0.01	4	0.02	26.17	29.81	26.58	20.03	8.70	5.24	2.98	0.01	7	0.01	25.97

Table 5:The optimal parameters of the economic statistical design of the FRS, VSI and VSIC Schemes with two warning limits for the case p = 2 and m = 25

	VSI scheme							FRS	VSIC scheme							
NO	k	w	h_1	h_2	n	ANF	E(L)	E(L)	k_1	k_2	w	h_1	h_2	n	ANF	E(L)
1	14.15	4.00	5.49	0.01	12	0.03	38.47	43.56	15.23	12.05	4.08	5.42	0.01	19	0.02	38.23
2	12.23	3.51	7.20	0.01	10	0.06	49.53	54.83	13.19	10.87	3.89	7.38	0.01	16	0.05	49.49
3	13.74	3.67	5.28	0.01	11	0.04	38.29	43.34	15.04	11.62	3.78	5.20	0.01	18	0.04	38.03
4	13.91	3.65	5.26	0.01	11	0.04	34.07	39.28	15.21	12.01	4.07	5.35	0.01	18	0.02	34.01
5	12.46	3.49	7.74	0.01	10	0.05	26.12	28.65	13.61	10.97	3.91	7.97	0.01	16	0.03	26.01
6	14.12	4.01	5.84	0.01	12	0.03	36.84	41.54	15.16	12.09	4.10	5.76	0.01	18	0.03	36.68
7	14.18	3.99	5.19	0.01	12	0.03	40.00	45.44	15.17	12.02	4.11	5.09	0.01	18	0.02	39.81
8	12.74	3.42	5.04	0.01	10	0.07	37.86	42.38	14.05	11.12	3.82	5.20	0.01	17	0.04	37.78
9	14.01	4.02	5.75	0.01	12	0.03	75.06	79.39	15.17	11.86	3.83	5.46	0.01	18	0.03	74.89
10	13.04	3.84	2.57	0.01	11	0.02	107.90	114.57	13,95	11.77	3.90	2.50	0.01	17	0.01	107.69
11	16.67	4.27	3.86	0.01	6	0.02	29.47	33.86	18.38	13.98	4.39	3.81	0.01	9	0.01	29.28
12	10.36	3.31	9.63	0.01	35	0.10	64.53	69.05	11.02	9.29	3.37	9.46	0.01	50	0.08	64.43
13	18.91	3.49	2.80	0.01	3	0.02	25.10	28.69	21.67	15.97	5.09	3.13	0.01	6	0.01	25.04

Table 6:The optimal parameters of the economic statistical design of the FRS, VSI and VSIC Schemes with two warning limits for the case p = 4 and m = 50

	VSI scheme						FRS	VSIC scheme								
NO	k	w	h_1	h_2	n	ANF	E(L)	E(L)	k_1	k_2	w	h_1	h_2	n	ANF	E(L)
1	18.00	6.78	5.95	0.01	14	0.03	41.21	46.00	19.01	15.85	6.87	5.81	0.01	22	0.03	41.01
2	16.20	6.65	8.19	0.01	13	0.05	53.57	58.21	16.97	14.45	6.70	8.12	0.01	19	0.04	53.47
3	17.57	6.42	5.76	0.01	13	0.04	41.11	45.80	18.75	15.51	6.92	5.88	0.01	22	0.04	39.96
4	18.01	6.78	5.92	0.01	14	0.03	36.92	41.47	19.02	15.73	6.90	5.85	0.01	22	0.03	36.68
5	16.44	6.62	8.81	0.01	13	0.05	28.01	30.23	17.02	14.58	6.73	8.64	0.01	19	0.03	27.85
6	17.96	6.79	6.33	0.01	14	0.03	39.43	43.85	18.94	15.68	6.89	6.25	0.01	22	0.02	39.32
7	18.03	6.77	5.63	0.01	14	0.04	42.89	48.03	18.95	15.71	6.91	5.57	0.01	22	0.03	42.69
8	17.03	6.91	5.92	0.01	14	0.05	40.76	44.81	17.50	14.69	6.65	5.68	0.01	20	0.05	40.34
9	17.59	6.42	6.04	0.01	13	0.04	77.46	81.55	18.71	15.62	6.89	6.10	0.01	21	0.03	77.27
10	17.01	7.03	2.90	0.01	14	0.02	113.98	119.36	17.29	15.47	7.10	2.85	0.01	21	0.02	113.76
11	20.59	7.19	4.17	0.01	7	0.02	21.21	35.41	22.01	17.71	7.34	4.1	0.01	11	0.02	31.09
12	13.93	6.07	10.74	0.01	43	0.11	70.33	74.61	14.11	12.58	5.99	10.52	0.01	50	0.09	70.31
13	22.67	7.09	3.16	0.01	4	0.02	26.17	29.81	24.67	19.90	8.77	3.39	0.01	7	0.01	26.20

(p = 4, m = 50). The results of applying one and two warning lines for the VSIC scheme in Table 3 and 5 (Table 4 and 6) indicate that the changes are not significant . Hence to overcome the difficulties in the application, it is recommended to apply the VSIC scheme with one warning line.

CONCLUDING REMARKS

In this paper, we have presented an economic design of T2 control chart with VSIC (two scales) while in-control process parameters (μ, Σ) are unknown. The cost model adopted in the present study is that of Costa and Rahim [3] and to find the optimal seven chart parameters (k1, k2, w1, w2, n, h1, h2) by using a genetic algorithm approach, the expected total cost is minimized. We have done meaningful and unbiased comparisons between VSIC (two and one scales), VSI and FRS chart and have shown that the VSIC chart is preferable to the VSI and FRS charts. We observe that the VSIC scheme with two warning lines is almost as economic as the VSIC scheme with one warning line. Hence because of the simple application and fewer parameters of the VSIC scheme with only one warning line, it is a better choice in practice.

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