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Exact Sequence in ϑ-nilpotent Groups

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Abstract: Let ϑ be a variety of groups defined by the set of laws V. We define the lower ϑ-verbal series, upper ϑ-marginal series and lower ϑ-marginal series of group G. A group G said to be ϑ-nilpotent group if there exists a series $1 = G_0 \le G_1 \le \cdots \le G_n = G$ where $G_i \trianglelefteq G$, $\frac{G_i}{G_{i-1}} \le V^*\left(\frac{G}{G_{i-1}}\right)$. In this note, we show if G be a ϑ-nilpotent group, then VP(G) is also ϑ-nilpotent and we also prove exact sequence about Bear-invariant of G.

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INTRODUCTION

Let V be a nonempty subset of a free group $F = \langle x_1, x_2, \ldots \rangle$ and ϑ be a variety of groups defined by set of laws V [4]. If G is a group with a normal subgroup N, then we define the Verbal subgroup, V(G) and the marginal subgroup, $V^*(G)$ and $[NV^*G]$ in the following:

$$V(G) = \langle \{ v(g_1,\ldots,g_n) \,;\, v \in V, g_i \in G, 1 \le i \le n\} \rangle$$

$$V^*(G) = \{ a \in G \,;\, v(g_1,\ldots,g_i a,\ldots,g_n) = v(g_1,\ldots,g_i,\ldots,g_n);\ v \in V,\ g_i \in G,\ 1 \le i \le n\}$$

$$[NV^*G] = \left\langle \left\{ \begin{array}{l} v(g_1,\ldots,g_i a,\ldots,g_n)\, v(g_1,\ldots,g_i,\ldots,g_n)^{-1}; \\ v \in V,\ g_i \in G,\ a \in N,\ 1 \le i \le n \end{array} \right\} \right\rangle$$

It is easily checked the verbal subgroup is fully invariant and the marginal subgroup is characteristic in G.

In the special case, where ϑ is the variety of groups defined by the set of laws $V = \{[x_1, x_2]\}$.

Then clearly ϑ is the variety of abelian groups and the verbal and the marginal subgroups of G are $V(G)=G'$, $V^*(G) = Z(G)$, respectively, if $N \trianglelefteq G$, then $[NV^*G] = [N,G]$.

If $V = \{[x_1,\ldots,x_{c+1}]\}$ is the nilpotent word, then ϑ is the variety of nilpotent groups of class at most c and $V(G)=\gamma_{c+1}(G)$, $V^*(G) = Z_c(G)$ and $[NV^*G] = [N,{}_cG]$.

The following lemma gives the properties of verbal and marginal subgroups of a given group, with respect to a given variety of groups ϑ [1].

Lemma 1.1: Let ϑ be a variety of groups and N be a normal subgroup of a given group G. Then the following statements hold :

(i) $V(V^*(G)) = 1$ and $V^*\left(\frac{G}{V(G)}\right) = \frac{G}{V(G)}$.

(ii) $V(G) = 1$ iff $V^*(G) = G$ iff $G \in \vartheta$.

(iii) $[NV^*G] = 1$ iff $N \le V^*(G)$.

(iv) $V\left(\frac{G}{N}\right) = \frac{V(G)N}{N}$ and $\frac{V^*(G)N}{N} \le V^*\left(\frac{G}{N}\right)$.

(v) $V(N) \le [NV^*G] \le N \cap V(G)$ inpatticular, $V(G)=[GV^*G]$.

(vi) if $N \cap V(G)=1$ then $N \le V^*(G)$ and $V^*\left(\frac{G}{N}\right) = \frac{V^*(G)}{N}$.

(vii) if $[G,N] \le V^*(G)$ then $[V(G),V^*(G)]=1$.

(viii) $V(G) \cap V^*(G)$, is contained in the Frattini subgroup of G.

Let ϑ be a variety of groups. we define the lower ϑ-verbal series of G to be

$$G = V^0(G) \ge V^1(G) \ge \ldots \ge V^n(G) \ge \ldots,$$

Where, for $n \ge 1$, $V^n(G) = V(V^{n-1}(G))$.

It is easy seen that $\frac{V^{n-1}(G)}{V^n(G)} \in \vartheta$.

The upper ϑ-marginal series of G to be

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$$1=V_0^*(G)\leq V_1^*(G)\leq\ldots\leq V_n^*(G)\leq\ldots,$$

Where, for

$$n\geq 1,\frac{V_n^*(G)}{V_{n-1}^*(G)}=V^*\left(\frac{G}{V_{n-1}^*(G)}\right).$$

The corresponding lower ϑ-marginal series of G

$$G=V_0(G)\geq V_1(G)\geq\ldots\geq V_n(G)\geq\ldots,$$

Where, for $n\geq 1$, $V_n(G)=[V_{n-1}(G)V^*G]$.

In the special case, where ϑ is the variety of abelian group, then $V^n(G)=G^{(n)}$ is the derived series and $V_n^*(G)=Z_n(G)$ is the upper central series and $V_n(G)=\gamma_{n+1}(G)$ is the lower central series of G.

Lemma 1.2. Let ϑ be a variety of groups and G be a group. Then the following properties hold :

(i) $V^i(V^j(G))=V^{i+j}(G), \forall i,j\geq 0$

(ii) $V_i^*\left(\frac{G}{V_j^*(G)}\right)=\frac{V_{i+j}^*(G)}{V_i^*(G)}, \forall i,j\geq 0$

(iii) $\frac{V_{i-1}(G)}{V_i(G)}\leq V^*\left(\frac{G}{V_i(G)}\right), \forall i\geq 1$

(iv) $V^*\left(\frac{V^{i-1}(G)}{V^i(G)}\right)=\frac{V^{i-1}(G)}{V^i(G)}, \forall\ i\geq 1$

(v) $V^i\left(\frac{G}{V^j(G)}\right)=\frac{V^i(G)}{V^j(G)}, \forall\ 0\leq i\leq j$

Proof: (i), (ii), (iii) and (iv) are clear from the definition and using induction. (v) By induction.

$$V^i\left(\frac{G}{V^j(G)}\right)=V(V^{i-1}\left(\frac{G}{V^j(G)}\right))=V\left(\frac{V^{i-1}(G)}{V^j(G)}\right)$$
$$=\frac{V(V^{i-1}(G))}{V^j(G)}=\frac{V^i(G)}{V^j(G)}.$$

H/K is said to be a ϑ-marginal factor of G if H and K are normal in G and $\frac{H}{K}\leq V^*\left(\frac{G}{K}\right)$.

Let $1\rightarrow N\overset{\mu}{\rightarrow}E\overset{\varepsilon}{\rightarrow}G\rightarrow 1$ (*) be an exact sequence of groups. Then we say that (*) is an extension of G by N.

A group G is said to be ϑ-nilpotent if there exists a series

$$G=G_0\geq G_1\geq\ldots\geq G_n=1\ (*)$$

where $G_i\trianglelefteq G$ and $\frac{G_{i-1}}{G_i}$ is a ϑ-marginal factor of G for i = 1,…,n. The Length of the shortest series (*) is the ϑ-nilpotent class of G.

The class of ϑ-nilpotent group is closed under the formation of subgroups, images and finite direct product. A group G is ϑ-nilpotent of class n, iff $V_n^*(G)=G$ or $V_{n+1}(G)=1$ (see [1]).

Let $1\rightarrow R\rightarrow F\rightarrow G\rightarrow 1$ be a free presentation of the group G. Then the Bear-invariant of G, with respect to the variety ϑ, defined by VM(G), is defined to be $\frac{R\cap V(F)}{[RV^*F]}$. we also denote the factor group $\frac{V(F)}{[RV^*F]}$ by VP(G). of course, if G is in ϑ, then VM(G) = VP(G). Inparticular, if successively ϑ is the variety of abelian or nilpotent group of class at most n (n>1), then the Bear-invariant of G will be $\frac{R\cap F'}{[R,F]}$ which by I.schure [5] is isomorphism to the schure-multiplicator of G, or $\frac{R\cap\gamma_{n+1}(G)}{[R,{}_nF]}$ (F repeated n times), respectively [3].

It is easy seen that the Bear-invariant of the group G with respect to the variety ϑ is always abelian and independent of the choice of the abelian and independent of the choice of the free presentation of G [2, 3].

MAIN RESULT

Lemma 2.1: let ϑ be a variety of groups defined by the set of laws Vand N, H are normal subgroups of a group G, then

$$\left[\frac{HN}{N}V^*\frac{G}{N}\right]=\frac{[HV^*G]N}{N}$$

Proof: suppose that $g_i\in G, h\in H, v\in V$ the proof is easy by using relation below

$$v(g_1N,\ldots,g_iNhN,\ldots,g_kN)v(g_1N,\ldots,g_iN,\ldots,g_kN)^{-1}$$
$$=v(g_1N,\ldots,g_ihN,\ldots,g_kN)v(g_1N,\ldots,g_kN)^{-1}$$
$$=v(g_1,\ldots,g_ih,\ldots,g_k)Nv(g_1,\ldots,g_k)^{-1}N$$
$$=v(g_1,\ldots,g_ih,\ldots,g_k)v(g_1,\ldots,g_k)^{-1}eN\in\frac{[HV^*G]N}{N}$$

all the relation are reflective thus by the upper relations the proof is competed.

Theorem 2.2: Let ϑ be a variety of groups and G be a ϑ-nilpotent group of class of $c\geq 2$. then

(i) $VM(G)\rightarrow VM\left(\frac{G}{V_c(G)}\right)\rightarrow V_c(G)\rightarrow 1$

(ii) $$VM(G)\rightarrow VM\left(\frac{G}{V^*_{c-1}(G)}\right)\rightarrow\frac{V^*_{c-1}(G)}{[V^*_{c-1}(G)V^*G]}\rightarrow\frac{G}{V(G)}\rightarrow\frac{G}{V(G)V^*_{c-1}(G)}\rightarrow 1$$

are exact sequence.

Proof: (i) consider the extension

$$1\rightarrow V_c(G)\xrightarrow{\mu}G\xrightarrow{\varepsilon}\frac{G}{V_c(G)}\rightarrow 1$$

now let $1\rightarrow R\rightarrow F\xrightarrow{\pi}G\rightarrow 1$ be a free presentation of G. Then $\varepsilon\pi:F\rightarrow\frac{G}{V_c(G)}$ is a free presentation for group $\frac{G}{V_c(G)}$. put ker $\varepsilon\pi = S$, then

$$VM(\frac{G}{V_c(G)}) = \frac{S\cap V(F)}{[SV^*F]}$$

and

$$VM(G) = \frac{R\cap V(F)}{[RV^*F]}$$

Clearly S is the inverse image of N under π, hence $R\subseteq S$ and

$$V_C(G)\cong\frac{S}{R}$$

define mapping

$$f_1: VM(G)\rightarrow VM\left(\frac{G}{V_c(G)}\right)$$

in natural way, one notes that its image is

$$\frac{(R\cap V(F))[SV^*F]}{[SV^*F]}$$

and the map

$$f_2: VM\left(\frac{G}{V_c(G)}\right)\rightarrow\frac{V_c(G)}{[V_c(G)V^*G]}$$

given by $x[SV^*F]\mapsto\pi(x)[V_c(G)V^*G]$. One can easily check that the image of f_1 is

$$\frac{(R\cap V(F))[SV^*F]}{[SV^*F]}$$

which is the same as the kernel of f_2. Thus we have exact sequence as follows

$$VM(G)\rightarrow VM\left(\frac{G}{V_c(G)}\right)\rightarrow\frac{V_c(G)}{[V_c(G)V^*G]}\rightarrow 1$$

Since G is ϑ-nilpotent of class c. So we have

$$[V_c(G)V^*G] = V_{c+1}(G) = 1$$

and the proof is completed.

(ii) Consider the extension

$$1\rightarrow V^*_{c-1}(G)\xrightarrow{\mu}G\xrightarrow{\varepsilon}\frac{G}{V^*_{c-1}(G)}\rightarrow 1$$

define mapping

$$f_1:\frac{G}{V(G)}\rightarrow\frac{G}{V(G)V^*_{c-1}(G)}$$

and

$$f_2:\frac{V^*_{c-1}(G)}{[V^*_{c-1}(G)V^*G]}\rightarrow\frac{G}{V(G)}$$

by

$$xV(G)\mapsto xV(G)V^*_{c-1}(G) \text{ and } x[V^*_{C-1}(G)V^*G]\mapsto xV(G)$$

respectively then the exactness at

$$\frac{G}{V(G)V^*_{c-1}(G)}$$

and

$$\frac{V^*_{c-1}(G)}{[V^*_{c-1}(G)V^*G]}$$

Can be easily checked.

Now, let $1\rightarrow R\rightarrow F\xrightarrow{\pi}G\rightarrow 1$ be a free presentation of G. Then $\varepsilon\pi:F\rightarrow\frac{G}{V^*_{c-1}(G)}$ is a free presentation for the group $\frac{G}{V^*_{c-1}(G)}$. Put ker $\varepsilon\pi = S$, then

$$VM\left(\frac{G}{V^*_{c-1}}\right) = \frac{S\cap V(F)}{[SV^*F]}$$

and

$$VM(G) = \frac{R\cap V(F)}{[RV^*F]}$$

clearly, S is the inverse image of N under π, hence $R\subseteq S$ and $N\cong\frac{S}{R}$.

Now we may define the map

$$f_3: VM\left(\frac{G}{V^*_{c-1}(G)}\right) \to \frac{V^*_{c-1}(G)}{[V^*_{c-1}(G)V^*G]}.$$

given by $x[SV^*F] \mapsto \pi(x)[V_{c-1}{}^*(G)V^*G]$. One can easily check that the image of f_3 is $\frac{V^*_{c-1}(G)\cap V(G)}{[V^*_{c-1}(G)V^*G]}$, which is the same as the kernel of f_2. Finally, we define the map

$$f_4: VM(G) \to VM\left(\frac{G}{V^*_{c-1}(G)}\right)$$

in the natural way. one notes that its image is

$$\frac{(R\cap V(F))[SV^*F]}{[SV^*F]}$$

On the other hand, we have Ker f_3 = im f_4. This gives the exactness of the sequence and the proof is completed.

Corollary 2.3: By the assumption of theorem 2.2, if the Bear-invariant of G is trivial, then

$$VM\left(\frac{G}{V_c(G)}\right) \cong V_c(G)$$

Proof: Using theorem 2.2 section (i) VM(G) be trivial then we have the following sequence

$$1 \to VM\left(\frac{G}{V_c(G)}\right) \to V_c(G) \to 1$$

this complete the proof.

Theorem 2.4: let ϑ be a variety of groups and G be a ϑ-nilpotent group, then VP(G) is ϑ-nilpotent.

Proof: Suppose that G be a ϑ-nilpotent group group of class c≥1. Such that G has free presentation $G \cong \frac{F}{R}$, we have $V_{c+1}(G) = 1$ it follows that $V_{c+1}\left(\frac{F}{R}\right) = 1$. Hence by using lemma 2.1 we have

$$\frac{V_{c+1}(F)R}{R} = 1$$

it implies that $V_{c+1}(F) \subseteq R(*)$. Now we show that VP(G) is ϑ-nilpotent, we have

$$V_{c+2}(VP(G)) = V_{c+2}\left(\frac{V(F)}{[RV^*F]}\right) = \frac{V_{c+2}(V(F))[RV^*F]}{[RV^*F]}$$
$$\subseteq \frac{V_{c+2}(F)[RV^*F]}{[RV^*F]} = \frac{[V_{c+2}(F)V^*F][RV^*F]}{[RV^*F]}$$

and by (*)we have $[V_{c+1}(F)V^*F] \subseteq [RV^*F]$ i.e $V_{c+2}(VP(G)) = 1$. This complete the proof.

Theorem 2.5: Let G be a ϑ-nilpotent group of class c≥2 and $G \cong \frac{F}{R}$ be a free presentation of G then;

(i) $|V_c(G)||VM(G)| = \left|VM\left(\frac{G}{V_c(G)}\right)\right|\left|\frac{[V_c(F)RV^*F]}{[RV^*F]}\right|$

(ii) $e(M(G)) \le e\left(VM\left(\frac{G}{V_c(G)}\right)\right)e\left(\frac{[V_c(F)RV^*F]}{[RV^*F]}\right)$

(iii) $d(VM(G)) \le d\left(VM\left(\frac{G}{V_c(G)}\right)\right) + d\left(\frac{[V_c(F)RV^*F]}{[RV^*F]}\right)$

Proof: (i) we can write

$$VM\left(\frac{G}{V_c(G)}\right) = VM\left(\frac{\frac{F}{R}}{\frac{V_c(F)R}{R}}\right)$$
$$\cong VM\left(\frac{F}{V_c(F)R}\right) \cong \frac{V(F)\cap V_c(F)R}{[V_c(F)RV^*F]}$$

Then We have

$$VM\left(\frac{G}{V_c(G)}\right) \cong \frac{(V(F)\cap R)V_c(F)}{[V_c(F)RV^*F]}$$

so

$$\frac{\frac{(V(F)\cap R)V_c(F)}{[RV^*F]}}{\frac{[V_c(F)RV^*F]}{[RV^*F]}} \cong \frac{(V(F)\cap R)V_c(F)}{[V_c(F)RV^*F]}$$

And

$$\left|\frac{(V(F)\cap R)V_c(F)}{[RV^*F]}\right| = \left|VM\left(\frac{G}{V_c(G)}\right)\right|\left|\frac{[V_c(F)RV^*F]}{[RV^*F]}\right|$$

this consequence that

$$\frac{\frac{(V(F)\cap R)V_c(F)}{[RV^*F]}}{\frac{(V(F)\cap R)}{[RV^*F]}} \cong \frac{(V(F)\cap R)V_c(F)}{(V(F)\cap R)} \cong \frac{V_c(F)}{(V(F)\cap R\cap V_c(F))}$$
$$\cong \frac{V_c(F)}{(R\cap V_c(F))} \cong \frac{V_c(F)R}{R} = V_c(G) \quad (*)$$

so we can write

$$\left|\frac{(V(F)\cap R)V_c(F)}{[RV^*F]}\right| = |V_c(G)||VM(G)|$$

(ii) We have

$$e(VM(G)) = e\left(\frac{(V(F)\cap R)}{[RV^* F]}\right) \le e\left(\frac{(V(F)\cap R)V_c(F)}{[RV^*F]}\right)$$

so by use of isomorphism theorem we have

$$e(VM(G)) \le e\left(VM\left(\frac{G}{V_c(G)}\right)\right) e\left(\frac{[V_c(F)RV^* F}{[RV^*F]}\right)$$

(iii) We have

$$d(VM(G)) = r(VM(G)) = r\left(\frac{V(F)\cap R}{[RV^*F]}\right) \le r\left(\frac{(V(F)\cap R)V_c(F)}{[RV^*F]}\right)$$

$$\le r\left(VM\left(\frac{G}{V_c(G)}\right)\right) + r\left(VM\left(\frac{[V_{c+1}(F)RV^*F]}{[RV^* F]}\right)\right)$$

And since $\frac{[V_{c+1}(F)RV^* F]}{[RV^*F]}$ is abelian because

$$\begin{aligned}
&[V_c(F)RV^*F]' = [[V_c(F)RV^*F],[V_c(F)RV^*F]] \\
&= [[V_c(F)V^*F][RV^*F],[V_c(F)V^*F][RV^*F]] \\
&= [[V_c(F)V^*F],[V_c(F)V^*F]][[V_c(F)V^*F],[RV^*F]] \\
&\quad [[RV^*F],[RV^*F]][[RV^*F],[V_c(F)V^*F]] \\
&\subseteq [[V_c(F)V^*F],V(F)][[V_{c-1}(F)V^*F],V(F)] \\
&\quad [RV^*F][V(F),[V_c(F)V^*F] \subseteq [RV^*F].
\end{aligned}$$

The proof is completed.

Theorem 2.6: The semi direct product of two groups that are ϑ-nilpotent is a ϑ-nilpotent group.

Proof: Let ϑ be an arbitrary variety of groups and H,K be two groups such that $\varphi: H \to Aut(K)$ be an arbitrary homomorphism and $G = H \ltimes_\varphi K$. Thus there exists two subgroups M and N such that and $G = MN, M\cap N = 1(*)$ so M and N are ϑ-nilpotent and by isomorphism theorem we have

$$\frac{G}{N} = \frac{MN}{N} \cong \frac{M}{N\cap M} \text{ and } \frac{G}{M} = \frac{MN}{M} \cong \frac{N}{M\cap N}$$

thus G/N, G/M are ϑ-nilpotent groups.

Becausee M and N are a ϑ-nilpotent groups. Now by lemma 2.1 we have

$$V_r\left(\frac{G}{N}\right) = \frac{V_r(G)N}{N} = 1_N$$

and

$$V_r\left(\frac{G}{M}\right) = \frac{V_r(G)M}{M} = 1_M .$$

Thus $V_r(G) \subseteq M\cap N$ and by (*) we have $V_r(G) = 1$ thus G is a ϑ-nilpotent group.

CONCLUSION

In this note, we show if G be a ϑ-nilpotent group, then VP(G) is also ϑ-nilpotent and we also prove exact sequence about Bear-invariant of G.

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