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# The Natural Lift Curves and the Geodesic Sprays for the Spherical Indicatrices of the Involutes of the Timelike Curve in Minkowski 3-Space

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**Abstract:** Let $\alpha$ be a timelike curve. In this situation its involute curve $\alpha^*$ must be a spacelike curve. In this paper, $(\alpha, \alpha^*)$ being the involute-evolute curve couple, the natural lift curves of the spherical indicatrices of tangent, principal normal, binormal vectors of the involute curve $\alpha^*$ have been given. Furthermore, some interesting results about the evolute curve $\alpha$ were obtained, depending on the assumption that the natural lift curves of the spherical indicatrices of the involute curve $\alpha^*$ should be the integral curve on the tangent bundle $T(S_1^2)$ or $T(H_0^2)$.

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**Key words:** Minkowski space . involüte-evolüte curve couple . geodesic spray . natural lift curve

## INTRODUCTION

C. Huygens, who is also known for his works in optics, discovered involutes while trying to build a more accurate clock. Involutes can be drawn easily by mechanical means: if a string is attached to a point on a curve, lying along the tangent to the curve at that point and is 'wrapped up' on to the curve, the locus of any point of the string is an involute of the given curve [4]. The original curve is called an evolute. A curve can have any number of involutes, thus a curve is an evolute of each of its involutes and an involute of its evolute. The normal to a curve is tangent to its evolute and the tangent to a curve is normal to its involutes. The evolute of a curve is also the envelope of the normal to the curve [9]. On the other hand, involute-evolute curve couple is a well known concept in the classical differential geometry [4, 7, 10]. In [5] authors have studied the natural lift curves of the spherical indicatrices of a given curve in the Euclidean 3-space $IR^3$. In differential geometry, especially the theory of space curves, the Darboux vector w can be expressed as $w = \tau t+\kappa b$ [7]. In [1, 2] authors have obtained the relationships between the Frenet frames of the timelike curve $\alpha$ and the spacelike involute curve $\alpha^*$ as depend on hyperbolic timelike angle $(\theta>0)$ between the unit tangent vector t and the normalisation of the Darboux vector $c = w/\|w\|$ in the Minkowski 3-space $IR_1^3$. In addition to this, the concepts of the natural lift and the geodesic sprays have been given by Thorpe [11]. Recently, Bilici *et al.* [3] have proposed the natural lift curves of the spherical indicatrices of the involutes of a given curve in $IR^3$.

In the present study, the natural lift curves of the spherical indicatrices of the involute curve $\alpha^*$ by taking the Minkowski 3-space $IR_1^3$ instead of $IR^3$ has been stated.

I hope these results will helpful to mathematicians who are specialized on mathematical modelling.

## PRELIMINARIES

Let M be a hypersurface in $IR_1^3$ equipped with a metric g, where the metric g means a symmetric non-degenerate (0,2) tensör field on M with constant signature. For a hypersurface M, let TM be the set $\cup\{T_p(M)\colon p\in M\}$ of all tangent vectors to M. A technicality: For each $p\in M$ replace $0\in T_p(M)$ by $0_p$ (other-wise the zero tangent vector is in every tangent space). Then each $v\in TM$ is in a unique $T_p(M)$ and the projection $\pi\colon TM\to M$ sends v to p. Thus $\pi^{-1}(p) = T_p(M)$. There is a natural way to make TM a manifold, called the tangent bundle of M.

A vector field $X\in\chi(M)$ is exactly a smooth section of TM, that is, a smooth function $X\colon M\to TM$ such that $\pi o X$ = identity.

Let M be a hypersurface in $IR_1^3$. A curve $\alpha\colon I\to M$ is an integral curve of $X\in\chi(M)$ provided $\dot{\alpha} = X_\alpha$; that is

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$$\frac{d}{dt}(\alpha(t)) = X(\alpha(t)) \text{ for all } t \in I \text{ [10]} \tag{1}$$

For any parametrized curve $\alpha: I \to M$, the parametrized curve

$$\bar{\alpha}: I \to TM$$

given by

$$\bar{\alpha}(t) = (\alpha(t), \dot{\alpha}(t)) = \dot{\alpha}(t)\big|_{\alpha(t)} \tag{2}$$

is called the natural lift of $\alpha$ on TM.
Thus, we can write

$$\frac{d\bar{\alpha}}{dt} = \frac{d}{dt}(\dot{\alpha}(t))\big|_{\alpha(t)} = D_{\dot{\alpha}(t)}\dot{\alpha}(t) \tag{3}$$

where D is the standart connection on $IR_1^3$.

For $v \in TM$, the smooth vector field $X \in \chi(M)$ defined by

$$X(v) = \varepsilon g(v, S(v))\xi\big|_{\alpha(t)}, \ \varepsilon = g(\xi, \xi) \tag{4}$$

is called the geodesic spray on the manifold TM, where $\xi$ is the unit normal vector field of M and S is the shape operator of M.

Let $\alpha$ be a unit speed timelike curve with curvature $\kappa$ and torsion $\tau$. Let Frenet frames of $\alpha$ be {t, n, b}. In this trihedron, t is timelike vector, n and b are spacelike vectors. For this vectors, we can write

$$t \times n = -b, \ n \times b = t, \ b \times t = -n$$

where $\times$ is the Lorentzian cross product [1] in space $IR_1^3$. In this situation, the Frenet formulas are given by

$$\dot{t} = \kappa n, \ \dot{n} = \kappa t - \tau b, \ \dot{b} = \tau n \ \text{[10]}$$

The Darboux vector for the timelike curve is given by

$$w = \tau t - \kappa b \ \text{[9]}$$

a) If $|\tau|<|\kappa|$, then w is a spacelike vector. In this situation, we can write

$$\begin{cases} \kappa = \|w\|\cosh\theta \\ \tau = \|w\|\sinh\theta \end{cases}, \ \|w\|^2 = g(w, w) = \kappa^2 - \tau^2$$

and

$$c = \frac{w}{\|w\|} = \sinh\theta \ t - \cosh\theta \ b$$

where c is unit vector of direction w.

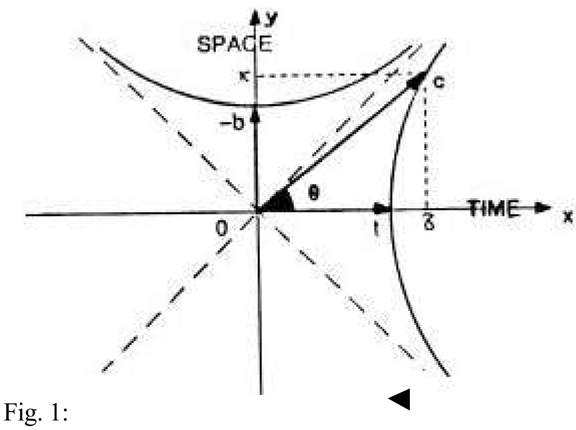


Fig. 1:

b) If $|\tau|<|\kappa|$, then w is a timelike vector. In this situation, we have

$$\begin{cases} \kappa = \|w\|\sinh\theta \\ \tau = \|w\|\cosh\theta \end{cases}, \ \|w\|^2 = -g(w, w) = \tau^2 - \kappa^2$$

and

$$c = \cosh\theta \ t - \sinh\theta \ b$$

where $\theta$ is hyperbolic timelike angle between t and c (Fig. 1).

From a) and b) $\frac{\tau}{\kappa} = \tanh\theta$ or $\frac{\tau}{\kappa} = \coth\theta$ and if $\theta$ = constant then $\alpha$ is a helix.

**Lemma 1:** The natural lift $\bar{\alpha}$ of the curve $\alpha$ is an integral curve of the geodesic spray X if and only if $\alpha$ is a geodesic on M.

**Proof ($\Rightarrow$):** Let $\bar{\alpha}$ be an integral curve of the geodesic spray X. Then we have

$$X(\bar{\alpha}(t)) = \frac{d}{dt}(\bar{\alpha}(t)) \tag{5}$$

Since X is a geodesic spray on TM, we have

$$X(\bar{\alpha}(t)) = \varepsilon g(\bar{\alpha}(t), S(\bar{\alpha}(t)))\xi\big|_{\alpha(t)} \tag{6}$$

From (2), (5) and (6) we get

$$\frac{d}{dt}\left(\dot{\alpha}(t)\big|_{\alpha(t)}\right) = \varepsilon g\left(\dot{\alpha}(t)\big|_{\alpha(t)}, S\left(\dot{\alpha}(t)\big|_{\alpha(t)}\right)\right)\xi\big|_{\alpha(t)} \tag{7}$$

Since the last equation is true for all $\alpha(t)$, using (3) we find that

$$D_{\dot{\alpha}(t)}\dot{\alpha}(t) = \varepsilon g(\dot{\alpha}(t), S(\dot{\alpha}(t)))\xi$$

Thus, from the last equation and Gauss formula we have

$$\overline{D}_{\dot{\alpha}(t)}\dot{\alpha}(t) = D_{\dot{\alpha}(t)}\dot{\alpha}(t) - \varepsilon\, g\big(\dot{\alpha}(t), S(\dot{\alpha}(t))\big)\xi = 0$$

where $\overline{D}$ is the Gauss connection on M. Hence we have seen that $\alpha$ is a geodesic on M.

($\Leftarrow$): Now assume that $\alpha$ be a geodesic on M. Then

$$\overline{D}_{\dot{\alpha}(t)}\dot{\alpha}(t) = 0 .$$

Hence, by the Gauss formula we have

$$D_{\dot{\alpha}(t)}\dot{\alpha}(t) - \varepsilon\, g\big(\dot{\alpha}(t), S(\dot{\alpha}(t))\big)\xi = 0$$

Since X is a geodesic spray, we can write

$$\frac{d}{dt}\Big(\dot{\alpha}((t))\big|_{\alpha(t)}\Big) - X\Big(\dot{\alpha}(t)\big|_{\alpha(t)}\Big) = 0$$

$$\frac{d}{dt}\Big(\dot{\alpha}(t)\big|_{\alpha(t)}\Big) = X\Big(\dot{\alpha}(t)\big|_{\alpha(t)}\Big)$$

From the equation (2), we find that

$$\frac{d}{dt}\big(\overline{\alpha}(t)\big) = X\big(\overline{\alpha}(t)\big)$$

**Lemma 2.** Let $(\alpha,\alpha^*)$ be the involute-evolute curve couple. The relations between the Frenet vectors of the curve couple as follow.

(1) If w is a spacelike vector($|\kappa|>|\tau|$), then

$$\begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh\theta & 0 & \sinh\theta \\ -\sinh\theta & 0 & \cosh\theta \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

(2) If w is a timelike vector($|\kappa|<|\tau|$), then

$$\begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh\theta & 0 & -\cosh\theta \\ -\cosh\theta & 0 & \sinh\theta \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \quad [1, 2]$$

## THE NATURAL LIFT OF THE SPHERICAL INDICATRIX OF TANGENT VECTOR OF THE INVOLUTE CURVE $\alpha^*$

Let $\alpha$ be a timelike curve with spacelike or timelike w. We will investigate how evolute curve $\alpha$ must be a curve satisfying the condition that $\overline{\alpha^*{}_{t^*}}$ is an integral curve of geodesic spray, where $\alpha^* t^*$ being the spherical indicatrix of tangent vector of involute curve $\alpha^*$, $\overline{\alpha^*{}_{t^*}}$ is the natural lift of the curve $\alpha^* t^*$.

If $\overline{\alpha^*{}_{t^*}}$ is an integral curve of the geodesic spray, then by means of Lemma1.

$$\overline{D}_{\dot{\alpha}^*_{t^*}}\dot{\alpha}^*_{t^*} = 0 \qquad (8)$$

where $\overline{D}$ is the connection of $S_1^2$ and the equation of the spherical indicatrix of tangent vector of the involute curve $\alpha^*$ is $\alpha^*_{t^*} = t^*$. Thus from Lemma 2. and (8) we have

$$\frac{\dot{\theta}}{\|w\|}\sinh\theta\, t - \frac{\dot{\theta}}{\|w\|}\cosh\theta\; b = 0$$

Because of {t, n, b} are linear independent, we have

$\theta = \text{constant}$, $\dfrac{\tau}{\kappa} =$ constant.

**Corollary 2.1:** If the curve $\alpha$ is a helix, then the spherical indicatrix $\alpha^* t^*$ of the involute curve $\alpha^*$ is a great circle on the Lorentzian unit sphere $S_1^2$. In this case, the natural lift $\overline{\alpha^*{}_{t^*}}$ of $\alpha^* t^*$ is an integral curve of the geodesic spray on the tangent bundle $T(S_1^2)$. In particular, if the evolute curve $\alpha$ is a timelike curve with timelike w, then the similar corollary is obtained

**Example 2.1:** Let

$$\alpha(\theta) = \left(\frac{2}{\sqrt{3}}\sinh\left(\frac{\theta}{\sqrt{3}}\right), \frac{2}{\sqrt{3}}\cosh\left(\frac{\theta}{\sqrt{3}}\right), \frac{\theta}{\sqrt{3}}\right)$$

be a unit speed time-like helix such that

$$t = \left(\frac{2}{\sqrt{3}}\cosh\left(\frac{\theta}{\sqrt{3}}\right), \frac{2}{\sqrt{3}}\text{sinhh}\left(\frac{\theta}{\sqrt{3}}\right), \frac{1}{\sqrt{3}}\right)$$

$$\kappa = \frac{2}{3} \text{ and } \tau = \frac{1}{3}$$

If $\alpha$ is a time-like curve then its involute curve is a space-like. In this situation, the involutes of the curve $\alpha$ can be given by the equation

The short calculations give the following equation of the spherical indicatrix of tangent vector of the involute curve $\alpha^*$

$$\alpha^*_{t^*} = t^* = \left(\sinh\left(\frac{\theta}{\sqrt{3}}\right), \cosh\left(\frac{\theta}{\sqrt{3}}\right), 0\right)$$

In this case $\alpha^* t^*$ is a timelike circle on $S_1^2$. Furthermore, we can write

$$g\left(t, t^*\right) = 0$$

## THE NATURAL LIFT OF THE SPHERICAL INDICATRIX OF PRINCIPAL NORMAL VECTORS OF INVOLUTE CURVE $\alpha^*$

Let $\alpha$ be a timelike curve with spacelike w. We will investigate in this section, how $\alpha$ must be a curve satisfying the condition that $\overline{\alpha^*_{n^*}}$ is an integral curve of the geodesic spray, where $\alpha^* n^*$ being the spherical indicatrix of principal normal vector of $\alpha^*$, $\overline{\alpha^*_{n^*}}$ is the natural lift of t he curve $\alpha^* n^*$.

If $\overline{\alpha^*_{n^*}}$ is an integral curve of the geodesic spray, then by means of Lemma1. we have

$$\overline{\overline{D}}_{\dot{\alpha}^*_{n^*}} \dot{\alpha}^*_{n^*} = 0 \ \ (\alpha^*_{n^*} = n^*)$$

From the Lemma 2. (1) we have,

$$, \ \sigma = \frac{\gamma_n}{k_n}$$

where $\overline{\overline{D}}$ is the connection of $H_0^2$, $\gamma_n$ and $k_n$ are the geodesic curvatures of the curve $\alpha$ with respect to $S_1^2$ and $IR_1^3$, respectively.

$$\left(\gamma_n = \frac{\dot{\theta}}{\|w\|}, k_n = \frac{1}{\|w\|}\sqrt{\left|\dot{\theta}^2 + \|w\|^2\right|}\right)$$

Since {t, n, b} are linear independent

and we get

$$k_n = \text{constant}, \ \gamma_n = \text{constant}$$

**Corollary 3.1:** If the geodesic curvatures of the evolute curve $\alpha$ with respect to $IR_1^3$ and $S_1^2$ are constant, then the spherical indicatrix $\alpha^* n^*$ is a great circle on the hyperbolic unit sphere $H_0^2$. In this case, the natural lift of $\alpha^* n^*$ is an integral curve of the geodesic spray on the tangent bundle . In particular, if the evolute curve $\alpha$ is a timelike curve with timelike w, then the similar corollary is obtained by taking $S_1^2$ instead of $H_0^2$.

## THE NATURAL LIFT OF THE SPHERICAL INDICATRIX OF THE BINORMAL VECTORS OF INVOLUTE CURVE $\alpha^*$

Let $\alpha$ be a timelike curve with spacelike or timelike w. We will investigate how $\alpha$ must be a curve satisfying the condition that is an integral curve of the geodesic spray, where $\alpha^* b^*$ being the spherical indicatrix of binormal vector of $\alpha^*$, is the natural lift of the curve $\alpha^* b^*$.

If is an integral curve of the geodesic spray, then by means of Lemma1.

that is,

Since {t, n, b} are linear independent, we have $\|w\| = 0$. Thus we get

$$\kappa = 0, \ \tau = 0$$

We may give the following corollary

**Corollary 4.1:** The spherical indicatrix $\alpha^* b^*$ of the involute curve $\alpha^*$ can't be a great circle on the hyperbolic unit sphere $H_0^2$, because, the evolute curve $\alpha$ whose curvature and torsion are equal to 0 is a straight line. In this case $(\alpha,\alpha^*)$ can't occur the involute-evolute curve couple. Thus, the natural lift of the curve $\alpha^* b^*$ can never be an integral curve of the geodesic spray on the tangent bundle .

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