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New Special Type Developable Surfaces in Terms of Focal Curve of Biharmonic Curve in the Heisenberg Group Heis^3

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Abstract: In this paper, we study new special type developable surface in terms of focal curve of biharmonic curve in the Heisenberg group Heis^3. We construct parametric equations of new special type developable surface. Finally, we show graphically the results obtained in main theorem.

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Key words: Heisenberg group · Biharmonic curve · Focal curve · Developable surface

INTRODUCTION

Surface representations are crucial to computer graphics, numerical simulation and computational geometry. Sampled representations, such as triangle meshes, have long served as simple, but effective, smooth surface approximations. The approximation of a smooth surfaces from a sampled geometric model, whether explicit or not, requires consistent notions of first-order and second-order differential geometric attributes, such as principal curvatures and principal directions. Typically, differential geometric properties are derived from surface vertices, mesh connectivity and, occasionally, by considering externally specified vertex normals.

The study and use of developable surfaces has a long history. Developable surfaces have natural applications in many areas of engineering and manufacturing. For instance, an aircraft designer uses them to design the airplane wings and a tinsmith uses them to connect two tubes of different shapes with planar segments of metal sheets. In computer graphics.

The aim of this paper is to study new special type developable surface in terms of focal curve of biharmonic curve in the Heisenberg group Heis^3.

Let (N, h) and (M, g) be Riemannian manifolds. Denote by R^N and R the Riemannian curvature tensors of N and M, respectively. We use the sign convention:

$$R^N(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, X, Y \in \Gamma(TN).$$

For a smooth map $\phi : N \to M$, the Levi-Civita connection ∇ of (N, h) induces a conncetion ∇^ϕ on the pull-back bundle

$$\phi^*TM = {}_{P\in N}\, T_{\phi(P)}M.$$

The section $\mathrm{T}(\phi)$: $= \mathrm{tr}\nabla^\phi d\phi$ is called the tension field of ϕ. A map ϕ is said to be harmonic if its tension field vanishes identically.

A smooth map ϕ: $N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2}\left|\mathrm{T}(\phi)\right|^2 dv_h.$$

The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_\phi \mathrm{T}(\phi) + \mathrm{tr}R(\mathrm{T}(\phi), d\phi)d\phi, \tag{1.1}$$

and called the bitension field of ϕ. The operator Δ_ϕ is the rough Laplacian acting on $\Gamma(\phi^*TM)$ defined by

$$\Delta_\phi := -\sum_{i=1}^{n}\left(\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla^N_{e_i}e_i}\right),$$

Where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study new special type developable surface in terms of focal curve of biharmonic curve in the Heisenberg group Heis^3. We construct parametric equations of new special type developable surface. Finally, We show graphically the results obtained in main theorem.

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Heisenberg Group Heis^3: Heisenberg group Heis^3 can be seen as the space R^3 endowed with the following multipilcation:

$$(\bar{x},\bar{y},\bar{z})(x,y,z)=(\bar{x}+x,\bar{y}+y,\bar{z}+z-\frac{1}{2}\bar{x}y+\frac{1}{2}x\bar{y}) \tag{2.1}$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.
The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz+\frac{y}{2}dx-\frac{x}{2}dy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$e_1=\frac{\partial}{\partial x}-\frac{y}{2}\frac{\partial}{\partial z},\ e_2=\frac{\partial}{\partial y}+\frac{x}{2}\frac{\partial}{\partial z},\ e_3=\frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[e_1,e_2]=e_3,[e_2,e_3]=[e_3,e_1]=0$$

with

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1.$$

We obtain

$$\begin{aligned}
&\nabla_{e_1}e_1=\nabla_{e_2}e_2=\nabla_{e_3}e_3=0,\\
&\nabla_{e_1}e_2=-\nabla_{e_2}e_1=\frac{1}{2}e_3,\\
&\nabla_{e_1}e_3=\nabla_{e_3}e_1=-\frac{1}{2}e_2,\\
&\nabla_{e_2}e_3=\nabla_{e_3}e_2=\frac{1}{2}e_1.
\end{aligned}$$

We adopt the following notation and sign convention for Riemannian curvature operator on Heis^3 defined by

$$R(X,Y)Z=-\nabla_X\nabla_Y Z+\nabla_Y\nabla_X Z+\nabla_{[X,Y]}Z,$$

While the Riemannian curvature tensor is given by

$$R(X,Y,Z,W)=g(R(X,Y)Z,W)$$

Where X, Y, Z, W are smooth vector fields on Heis^3

The components $\{R_{ijkl}\}$ of R relative to $\{e_1,e_2,e_3\}$ are defined by

$$g(R(e_i,e_j)e_k,e_l)=R_{ijkl}.$$

The non-vanishing components of the above tensor fields are

$$\begin{aligned}
&R_{121}=-\frac{3}{4}e_2,\quad R_{131}=\frac{1}{4}e_3,\quad R_{122}=\frac{3}{4}e_1,\\
&R_{232}=\frac{1}{4}e_3,\quad R_{133}=-\frac{1}{4}e_1,\quad R_{233}=-\frac{1}{4}e_2,
\end{aligned}$$

and

$$R_{1212}=-\frac{3}{4},\quad R_{1313}=R_{2323}=\frac{1}{4}. \tag{2.3}$$

Biharmonic Curves in the Heisenberg Group Heis^3: Let I be an open interval and $\gamma: I\to Heis^3$ be a curve parametrized by arc length on Heisenberg group Heis^3. Putting $t=\gamma'$, we can write the tension field of γ as $\tau(\gamma)=\nabla_{\gamma'}\gamma'$ and the biharmonic map equation (1.1) reduces to

$$\nabla_{\mathbf{t}}^3\mathbf{t}+R(\mathbf{t},\nabla_{\mathbf{t}}\mathbf{t})\mathbf{t}=0. \tag{3.1}$$

A successful key to study the geometry of a curve is to use the Frenet frames along the curve, which is recalled in the following.

Let $\gamma: I\to Heis^3$ be a curve on $Heis^3$ parametrized by arc length. Let $\{t, n_1, n_2\}$ be the Frenet frame fields tangent to Heis^3 along γ defined as follows: t is the unit vector field γ' tangent to γ, n_1 is the unit vector field in the direction of $\nabla_t t$ (normal to γ) and b is chosen so that $\{t, n_1, n_2\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}
&\nabla_t t=\kappa t,\\
&\nabla_t n_1=-\kappa t-\tau n_2,\\
&\nabla_t n_2=\tau n_1
\end{aligned} \tag{3.2}$$

Where $\kappa=|\nabla_T T|$ is the curvature of γ and τ is its torsion. With respect to the orthonormal basis $\{e_1,e_2,e_3\}$, we can write

$$\begin{aligned}
&\mathbf{t}=t_1\mathbf{e}_1+t_2\mathbf{e}_2+t_3\mathbf{e}_3,\\
&\mathbf{n}_1=n_1^1\mathbf{e}_1+n_1^2\mathbf{e}_2+n_1^3\mathbf{e}_3,\\
&\mathbf{n}_2=\mathbf{t}\times\mathbf{n}_1=n_2^1\mathbf{e}_1+n_2^2\mathbf{e}_2+n_2^3\mathbf{e}_3.
\end{aligned}$$

Theorem 3.1: *Let $\gamma: I\to Heis^3$ be a non-geodesic curve on Heis^3 parametrized by arc length. Then, γ is a non-geodesic biharmonic curve if and only if*

$$\begin{aligned}
&\kappa=\text{constant}\neq 0,\\
&\kappa^2+\tau^2=\frac{1}{4}-\left(n_2^3\right)^2,\\
&\tau'=0.
\end{aligned} \tag{3.3}$$

Developable Surfaces Associated with a Focal Curve of Biharmonic Curve in Heisenberg Group Heis^3: For a unit speed curve γ, the curve consisting of the centers of the osculating spheres of γ is called the parametrized focal curve of γ. The hyperplanes normal to γ at a point consist of the set of centers of all spheres tangent to γ at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by C_γ, we can write.

$$C_\gamma(s) = (\gamma + c_1 \mathrm{n}_1 + c_2 \mathrm{n}_2)(s) \tag{4.1}$$

Where the coefficients c_1, c_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1 = \frac{1}{\kappa}, c_2 = \frac{c_1'}{\tau}, \kappa \neq 0, \tau \neq 0. \tag{4.2}$$

Lemma 4.1: *Let $\gamma : I \to Heis^3$ be a unit speed biharmonic curve and and C_γ its focal curve on $Heis^3$. Then,*

$$c_1 = \frac{1}{\kappa} = \text{constant and } c_2 = 0. \tag{4.3}$$

Proof: Using (3.3) and (4.2), we get (4.3).

Lemma 4.2: *Let $\gamma : I \to Heis^3$ be a unit speed biharmonic curve and and C_γ its focal curve on $Heis^3$. Then,*

$$C_\gamma(s) = (\gamma + c_1 \mathrm{n}_1)(s) \tag{4.4}$$

On the other hand, a ruled surface in Heis^3 is (locally) the map $\Omega_{(\gamma,\delta)}$ defined by

$$\Omega_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s),$$

Where $\gamma : I \to Heis^3$, $\delta : I \to Heis^3 \setminus \{0\}$ are smooth mappings and I is an open interval or the unit circle S^1. We call the base curve and the director curve. The straight lines $u \to \gamma(s) + u\delta(s)$ are called rulings of $\Omega_{(\gamma,\delta)}$.

We now consider a special type of ruled surface, which has been studied for over a century, the developable surface. Informally, these are surfaces which can be attened onto a plane without distortion, so are a transformation (e.g. folding or bending) of a plane in Heis^3. It is this fundamental property which has long ensured their useful application in engineering and manufacturing. More recently, their use has spread to the computer sciences, in computer-aided design; their isometric properties make them ideal primitives for texture mapping.

Definition 4.3: *A smooth surface $\hat{o}_{(\gamma,\delta)}$ is called a developable surface if its Gaussian curvature K vanishes everywhere on the surface.*

Definition 4.4: *Let $\gamma : I \to Heis^3$ be a unit speed curve. We define the following developable surface*

Where $C_\gamma(s)$ is focal curve.

Theorem 4.5: *Let $\gamma : I \to Heis^3$ be a unit speed biharmonic curve and and C_γ its focal curve on $Heis^3$. Then, the parametric equations of $\Omega_{(\gamma,\delta)}$ are*

$$\begin{aligned}
x_{\left(C_\gamma,\gamma'\right)}(s,u) &= \frac{c_1}{\kappa}\sin\varphi(\cos\varphi - \Lambda)\sin[\Lambda s + \rho] \\
&+ \frac{1}{\Lambda}\sin\varphi\sin[\Re s + \rho] + u\sin\varphi\cos[\Lambda s + \rho] + a_1, \\
y_{\left(C_\gamma,\gamma'\right)}(s,u) &= -\frac{c_1}{\kappa}\sin\varphi(\cos\varphi - \Lambda)\cos[\Lambda s + \rho] \\
&- \frac{1}{\Lambda}\sin\varphi\cos[\Lambda s + \rho] + u\sin\varphi\sin[\Lambda s + \rho] + a_2, \\
z_{\left(C_\gamma,\gamma'\right)}(s,u) &= (\cos\varphi + \frac{1}{4\Lambda}\sin^2\varphi)(s+u) \\
&- \frac{c_1}{\kappa}\sin\varphi(\cos\varphi - \Lambda)\sin[\Lambda s + \rho]\left(\frac{1}{2}q_3 s + \frac{1}{2}q_4\right) \\
&- \frac{c_1}{\kappa}\sin\varphi(\cos\varphi - \Re)\cos[\Lambda s + \rho]\left(\frac{1}{2}q_1 s + \frac{1}{2}q_2\right) + a_3,
\end{aligned} \tag{4.5}$$

where $\rho, q_1, q_2, q_3, q_4, a_1, a_2, a_3,$ are constants of integration and $\Lambda = \frac{\cos\varphi \pm \sqrt{5(\cos\varphi)^2 - 4}}{2}$.

Proof: The covariant derivative of the vector field t is:

$$\nabla_{\mathbf{t}}\mathbf{t} = (t_1' + t_2 t_3)\mathbf{e}_1 + (t_2' - t_1 t_3)\mathbf{e}_2 + t_3'\mathbf{e}_3. \tag{4.6}$$

Thus using Theorem 3.2, we have

$$\mathrm{t} = (\sin\varphi\cos[\Lambda s + \rho]\mathrm{e}_1 + \sin\varphi\sin[\Lambda s + \rho]\mathrm{e}_2 + \cos\varphi\mathrm{e}_3 \tag{4.7}$$

Where $\Lambda = \frac{\cos\varphi \pm \sqrt{5(\cos\varphi)^2 - 4}}{2}$.

Using (2.2) in (4.3), we obtain

$$\mathbf{t} = (\sin\varphi\cos[\Lambda s + \rho], \sin\varphi\sin[\Lambda s + \rho],$$
$$\cos\varphi - \frac{1}{2}y(s)\sin\varphi\cos[\Lambda s + \rho] + \frac{1}{2}x(s)\sin\varphi\sin[\Lambda s + \rho]).$$

From (2.2), we get

$$\mathbf{t} = (\sin\varphi\cos[\Lambda s+\rho], \sin\varphi\sin[\Lambda s+\rho], \cos\varphi + \frac{1}{2\Lambda}\sin^2\varphi). \tag{4.8}$$

From (4.6) and (4.7), we get

$$\nabla_{\mathbf{t}}\mathbf{t} = \sin\varphi(\cos\varphi - \Lambda)(\sin[\Lambda s+\rho]\mathbf{e}_1 - \cos[\Lambda s+\rho]\mathbf{e}_2).$$

By the use of Frenet formulas, we get

$$\begin{aligned}\mathbf{n}_1 &= \frac{1}{\kappa}\nabla_{\mathbf{t}}\mathbf{t} \\ &= \frac{1}{\kappa}[\sin\varphi(\cos\varphi-\Lambda)(\sin[\Lambda s+\rho]\mathbf{e}_1 - \cos[\Lambda s+\rho]\mathbf{e}_2)].\end{aligned} \tag{4.9}$$

Substituting (2.2) in (4.9), we have

$$\begin{aligned}\mathbf{n}_1 = \frac{1}{\kappa}\sin\varphi(\cos\varphi-\Lambda)(&\sin[\Lambda s+\rho], -\cos[\Lambda s+\rho], \\ &-\sin[\Lambda s+\rho]\left(\frac{1}{2}q_3 s+\frac{1}{2}q_4\right) - \cos[\Lambda s+\rho]\left(\frac{1}{2}q_1 s+\frac{1}{2}q_2\right)),\end{aligned} \tag{4.10}$$

Where q_1, q_2, q_3, q_4, are constants of integration.

We substitute (4.8) and (4.10) into Definition 4.4, we get (4.5). The proof is completed.

Using Mathematica in Theorem 4.5 for different constant, yields

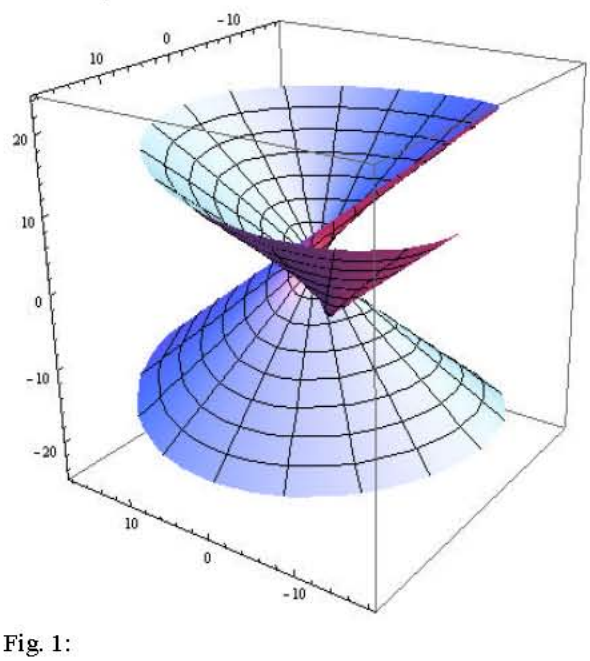

Fig. 1:

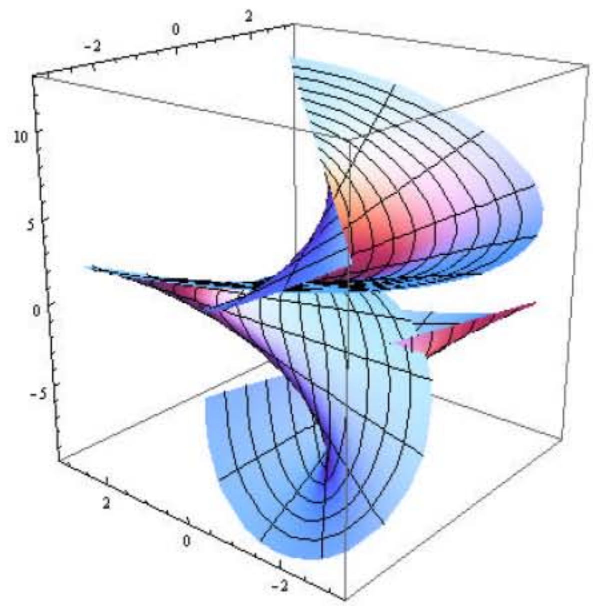

Fig. 2:

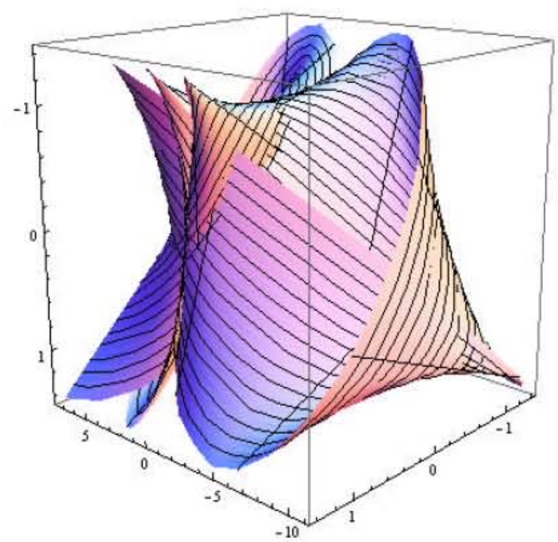

Fig. 3:

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