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Homotopy Perturbation and Taylor Series for Volterra Integral Equations of the Second Kind

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Abstract: In this paper, coupling homotopy perturbation method and Taylor series for solving linear and non-linear Volterra integral equations of the second kind, has been introduced. Comparisons of the results of applying new technique and classical HPM reveal the new technique is very effective and convenient.

Key words: Coupling homotopy perturbation method · Taylor series · Volterra integral equations

INTRODUCTION

Homotopy perturbation method has been used by many mathematicians and engineers to solve various kinds of functional equations. In this method the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solutions [1-5]. Homotopy perturbation continuously deforms the difficult equation under study into a simple equation, easy to solve. In recent years the application of homotopy perturbation theory has appeared in many researches [7-16].

In this paper we introduce coupling homotopy perturbation method and Taylor series for solving Volterra integral equations of the second kind and comparisons are made between the new technique and the classical homotopy perturbation.

Consider the Following Volterra Integral Equation:

$$u(t) = f(t) + \int_a^t k(s,t)u(s)ds. \tag{1}$$

Let's consider

$$L(u) = u(t) - f(t) - \int_a^t k(s,t)u(s)ds = 0, \tag{2}$$

and it's with solution, $u(t) = \varphi(t)$ By the homotopy technique, the homotopy can be defined by

$$H(u,p) = (1-p)F(u) + pL(u) = 0. \tag{3}$$

Where $F(u)$ is a functional operator and let's u_0 be the solution of $F(u) = 0$ which can be obtained easily. Obviously, from Eq. (3) results in

$$H(u,0) = F(u), \quad H(u,1) = L(u),$$

and continuously trace an implicitly defined curve from a starting point $H(u_0,0)$ to a solution $H(\varphi 1,)$. The embedding parameter p monotonically increases from zero to unity as the trivial problem $F(u) = 0$ continuously deformed to the original problem $L(\varphi) = 0$.

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The homotopy perturbation method considers the solution as a power series, say

$$u = u_0 + pu_1 + p^2u_2 + \cdots. \tag{4}$$

Where p is called an embedding parameters [6]. Substituting (4) into (3) and equating the terms with identical power of p we obtain

$$\begin{aligned}
&p^0 : u_0(t) = f(t),\\
&p^1 : u_1(t) = \int_a^t k(s,t)u_0(s)ds,\\
&p^2 : u_2(t) = \int_a^t k(s,t)u_1(s)ds,\\
&\vdots\\
&p^j : u_j(t) = \int_a^t k(s,t)u_{j-1}(s)ds.
\end{aligned}$$

These terms can be used to construct the solution, for $p = 1$, in (4).

Coupling Homotopy Perturbation Method and Taylor Series: To accelerate the convergence of homotopy perturbation method, when it is used for Volterra integral equations, if the kernel $k(s,t)$ is separable, say $k(s,t) = k_1(s)k_2(t)$ and functions $k_1(s)$, $k_2(t)$ and $f(t)$ are analytic suggested to be replaced by their Taylor series forms

$$f(t) = \sum_{i=0}^{\infty} f_i(t), \quad k(s,t) = k_1(s)k_2(t) = \sum_{i=0}^{\infty} k_{1i}(s) \sum_{i=0}^{\infty} k_{2i}(t). \tag{5}$$

With $f_i(t) = \dfrac{f^{(i)}(t_0)(t-t_0)^i}{i!}$, $k_{1i}(t) = \dfrac{k_1^{(i)}(t_0)(t-t_0)^i}{i!}$ and $k_{2i}(s) = \dfrac{k_2^{(i)}(s_0)(s-s_0)^i}{i!}$, respectively.

Substitution Eqs. (5) into Eq. (1) results in

$$L(u) = u(t) - \sum_{i=0}^{\infty} f_i(t) - \int_a^x \sum_{i=0}^{\infty} k_{1i}(s) \sum_{i=0}^{\infty} k_{2i}(t)u(s)ds = 0.$$

The following homotopy can be constructed

$$H(u,p) = u(t) - \sum_{i=0}^{\infty} f_i(t)p^i - p\int_a^x \sum_{i=0}^{\infty} k_{1i}(s)p^i \sum_{i=0}^{\infty} k_{2i}(t)p^i u(s)ds = 0. \tag{6}$$

Substituting (4) into (6) and equating the coefficients of the terms with identical powers of p, components of series solution (4) will be obtained.

This technique is simple and very effective tool which usually leads to the exact solutions. This method can be used for problems that the homotopy perturbation method doesn't work.

Numerical Example: In this part four examples are provided. These examples are considered to illustrate ability and reliability of the new technique.

Example 1: Consider the following Volterra integral equation

$$y(x) = f(x) - \int_0^x e^{-(t-x)} y(t)dt, \tag{7}$$

Where

$$f(x) = \cosh x,$$

With the exact solution $y(x) = e^{-x}$.

Homotopy Perturbation Method:
Consider the following homotopy

$$y(x) = f(x) - p\int_0^x e^{-(t-x)} y(t)dt. \tag{8}$$

Substituting (4) into (8) and equating the coefficients of the terms with identical powers of p, the following terms will be obtained

$$\begin{aligned}
&p^0 : y_0(x) = f(x) = \cosh x,\\
&p^1 : y_1(x) = -\int_0^x e^{-(t-x)} y_0(t)dt, \Rightarrow y_1(x) = -\frac{1}{2}\sinh x - \frac{1}{2}x\sinh x - \frac{1}{2}x\cosh x,\\
&p^2 : y_2(x) = -\int_0^x e^{-(t-x)} y_1(t)dt, \Rightarrow y_2(x) = -\frac{1}{4}\sinh x + \frac{1}{4}x\sinh x + \frac{1}{4}x^2\sinh x + \frac{1}{4}x\cosh x + \frac{1}{4}x^2\cosh x,\\
&\vdots\\
&p^j : y_j(x) = -\int_0^x e^{-(t-x)} y_{j-1}(t)dt.
\end{aligned}$$

Then the series solution by standard HPM will be as follows

$$y(x) = \sum_{i=0}^{\infty} y_i(x) = \cosh x - \frac{1}{2}\sinh x - \frac{1}{2}x\sinh x - \frac{1}{2}x\cosh x + -\frac{1}{4}\sinh x + \frac{1}{4}x\sinh x + \frac{1}{4}x^2\sinh x + \cdots.$$

The New Technique:

Taylor series of $fd(x)$ and $k(x,t)$ will be used

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^n}{n!} + \frac{(-x)^n}{n!}\right), \qquad k(x,t) = \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}.$$

Substitution of these series in the equation (7), the following homotopy can be constructed

$$y(x) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^n}{n!} + \frac{(-x)^n}{n!}\right)p^n - p\int_0^x \sum_{n=0}^{\infty}\frac{x^n}{n!}p^n \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}p^n y(t)dt. \tag{9}$$

Substituting (4) into (9) and equating the terms with identical powers of p, leads to

$$\begin{aligned}
&p^0 : y_0(x) = 1,\\
&p^1 : y_1(x) = -\int_0^x y_0(t)dt, \Rightarrow y_1(x) = -x,\\
&p^2 : y_2(x) = \frac{1}{2}x^2 - \int_0^x \left(-ty_0(t) + xy_0(t) + y_1(t)\right)dt \Rightarrow y_2(x) = \frac{1}{2}x^2,\\
&\vdots\\
&p^j : y_j(x) = \frac{1}{2}\left(\frac{x^j}{j!} + \frac{(-x)^j}{j!}\right) - \int_0^x \left(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\frac{x^i}{i!}\frac{t^k}{k!}y_{j-k-i-1}(t)\right)dt.
\end{aligned}$$

The series form of the solution is given by

$$y(x)=\sum_{i=0}^{\infty}y_i(x)=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4+\ldots,$$

and hence, $y(x) = e^{-x}$ which is the exact solution of Example 1.

Example 3: Consider the following nonlinear Volterra integral equation

$$y(x)=f(x)+\frac{3}{2}\int_0^x \cos(t-x)y^2(t)dt, \tag{10}$$

Where

$$f(x)=\frac{1}{2}\sin 2x,$$

With exact solution $y(x) = \sin x$

Homotopy Perturbation Method: Using homotopy perturbation method, leads to

$$y(x)=f(x)+\frac{3}{2}p\int_0^x \cos(t-x)y^2(t)dt. \tag{11}$$

Substituting (4) into (11) and equating the coefficients of the terms with identical powers of p, the following terms will be achieved

$$\begin{aligned}
&p^0: y_0(x)=f(x)=\frac{1}{2}\sin 2x,\\
&p^1: y_1(x)=\frac{3}{2}\int_0^x \cos(t-x)y_0{}^2(t)dt, \Rightarrow y_1(x)=\frac{1}{5}\sin x-\frac{1}{20}\sin 4x,\\
&\vdots\\
&p^j: y_j(x)=\frac{3}{2}\int_0^x \cos(t-x)\left(\sum_{k=0}^{j-1}y_k(t)y_{j-k-1}(t)\right)dt.
\end{aligned}$$

Then the series solution, by the homotopy perturbation method, is as follows

$$y(x)=\sum_{i=0}^{\infty}y_i(x)=\frac{1}{2}\sin 2x+\frac{1}{5}\sin x+\frac{589}{5600}\sin x-\frac{9}{160}\sin 3x+\frac{3}{40}x\cos x-\frac{1}{40}\sin 2x+\frac{9}{1400}\sin 6x+\cdots.$$

The New Technique: We use the Taylor series for $f(x)$ and $k(k,\ t)$

$$\begin{aligned}
f(x)&=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!},\\
k(x,t)&=\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

And construct the following homotopy, after substitution of Taylor series in the equation (10)

$$y(x)=\frac{1}{2}\sum_{n=0}^{\infty}\left((-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}\right)p^n+p\int_0^x\begin{pmatrix}\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}p^n\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}p^n\\+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!}p^n\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}p^n\end{pmatrix}y^2(t)dt. \tag{12}$$

Substituting (4) into (15) and equating of the terms with identical powers of p, gives

$$p^0: y_0(x)=x,$$

$$p^1: y_1(x)=-\frac{2}{3}x^3+\frac{3}{2}\int_0^x\left(y_0{}^2(t)+xty_0{}^2(t)\right)dt,\Rightarrow y_1(x)=\ -\frac{1}{6}x^3+\frac{3}{8}x^5,$$

$$p^2: y_2(x)=\frac{2}{15}x^5+\frac{3}{2}\int_0^x\left(-\frac{1}{2}t^2y_0{}^2(t)-\frac{1}{2}x^2y_0{}^2(t)+2y_1(t)y_0(t)-\frac{1}{6}t^3y_0{}^2(t)-\frac{1}{6}x^3y_0{}^2(t)+2xty_1(t)y_0(t)\right)dt,$$

$$\Rightarrow y_2(x)=\ -\frac{11}{30}x^5+\frac{9}{64}x^9-\frac{3}{112}x^7,$$

$$p^j: y_j(x)=(-1)^j\frac{(2x)^{2j+1}}{(2j+1)!}+\frac{3}{2}\int_0^x\begin{pmatrix}\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\sum_{l=0}^{j-i-k-1}(-1)^i\frac{t^{2i}}{(2i)!}(-1)^k\frac{x^{2k}}{(2k)!}y_l(t)y_{j-l-k-i-1}(t)\\+\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\sum_{l=0}^{j-i-k-1}(-1)^i\frac{t^{(2i+1)}}{(2i+1)!}(-1)^k\frac{x^{(2k+1)}}{(2k+1)!}y_l(t)y_{j-l-k-i-1}(t)\end{pmatrix}dt.$$

Therefore the solution of Example 3 can be readily presented by

$$y(x)=\sum_{i=0}^{\infty}y_i(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\ldots,$$

With the summation, $y(x)$ which is the exact solution.

CONCLUSION

In this paper, coupling homotopy perturbation method and Taylor series, for solving Volterra integral equations of the second kind, is presented successfully. This new idea is based on the series forms of the function $f(t)$ and the kernel $K(x, t)$. So it is necessary to mention that this approach can be used when $f(t)$, $K(x, t)$ are analytic. The most important note which is worth to mention is that this procedure leads, almost, to the exact solution for both linear and nonlinear equations. The computations associated with the examples in this paper were performed using the package maple 13.

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