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Explicit Solution of the Space-Time Fractional Klein-Gordon Equation of Distributed Order via the Fox H-Functions

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Abstract: In this article, the space-time fractional Klein-Gordon equation of distributed order is introduced and some aspects of particular cases of this equation such as symmetry and single order cases are expressed. Also, using the Fourier, Laplace and Mellin integral transforms fundamental solutions of these equations are obtained through the Fox H-functions.

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INTRODUCTION

The fractional differential operator of distributed order

$$D^{\beta}_{\text{do}} = \int_l^u b(\beta) \frac{d^{\beta}}{dt^{\beta}} d\beta, \quad u > l \geq 0, b(\beta) \geq 0 \tag{1}$$

is a generalization of the single order $D^{\beta}_{\text{so}} = \frac{d^{\beta}}{dt^{\beta}}$ which by considering a continuous or discrete distribution of fractional derivative is obtained. The idea of fractional derivative of distributed order was stated by Caputo [1] and was developed by Caputo himself [2, 3]and Bagley and Torvik [4] later. Other researchers used this idea and appeared interesting reviews to describe the related mathematical models of partial fractional differential equation of distributed order.

For example Mainardi [5, 6], Chechkin *et al.* [7-9], Umarov *et al.* [10], Kochubei [11], Sun *et al.* [12], Aghili [13] investigated on some linear distributed-order boundary value problems of form

$$\int_0^m b(\beta) D^{\beta} u(x,t) d\beta = B(D) u(x,t) \quad D = \frac{d}{dx}, t > 0, x \in \mathbb{R} \tag{2}$$

With pseudo-differential operator $B(D)$ and the Cauchy conditions

$$\frac{\partial^k}{\partial t^k} u(x,0^+) = f_k(x) \quad k = 0,1,..,m-1. \tag{3}$$

In particular cases the characteristics of time-fractional or space-time fractional diffusion equation of distributed order were studied for treatises in the sub, normal and super diffusions [14-16].

Now, in this paper in distributed-order equations class in section 2 we introduce the space-time fractional Klein-Gordon equation of distributed order in the Riesz-Feller and Caputo senses and focus on mathematical aspects and technical approaches to find the explicit solutions of this equation. In this regard, we choose the operational calculus scope to find the fundamental solutions via the Fox H-functions.

In this sense, as a special case in section 3 we obtain the solution of symmetric space-time fractional Klein-Gordon equation of distributed order. The Mellin transform is the alternative tool to change the solution into the Mellin-Barens integral and construction of the Fox H-functions.

In section 4, as other special case we study the space-time fractional wave equation of single order and show the explicit solution via the Fox H-functions. Finally, in section 5 the main conclusions are drawn and for convenience appendices are included to our notations of fractional calculus and introducing the Fox H-function.

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The Space-Time Fractional Klein-Gordon Equation of Distributed Order: The following equation is called the space -time fractional Klein-Gordon equation of distributed order

$$\int_1^2 b(\beta)[{}_t^C D_{0^+}^{\beta} u(x,t)]d\beta - c^2[{}_x^{RF} D_{\theta}^{\alpha} u(x,t)] + d^2 u = q(x,t), \tag{1}$$

$$u(x,0) = f(x), u_t(x,0) = g(x), \lim_{|x|\to\infty} u(x,t) = 0,, \tag{2}$$

$$t > 0, x, c, d \in \mathbb{R}, b(\beta) \ge 0, \int_1^2 b(\beta)d\beta = 1. \tag{3}$$

With the order-density function $b(\beta)$ and the Cauchy type initial and boundary conditions. The parameters α,β,θ are the real and are restricted to

$$0 < \alpha \le 2, 1 < \beta \le 2, |\theta| \le \min\{\alpha 2 - \alpha\} \tag{4}$$

and ${}_x^{RF} D_{\theta}^{\alpha}, {}_t^C D_{0^+}^{\beta}$ are integrao-differential operators, the Riesz-Feller space-fractional derivative order α and asymmetry θ and the Caputo time-fractional derivative of order β, respectively, see (A.1), (A.3).

In order to solve the equation (??), we extend the approach by Naber [17] to find a general representation of the fundamental solution related to a generic order-density function $b(\beta)$. In this respect, by applying the Laplace transform with respect to t (A.2)

$$L\{{}_t^C D_{0^+}^{\beta} u(x,t); s\} = s^{\beta}\tilde{u}(x,s) - s^{\beta-1}u(x,0^+) - s^{\beta-2}u_t(x,0^+), \quad s \in \mathbb{C} \tag{5}$$

and the Fourier transform of the Riesz-Feller fractional derivative with respect to x (A.4)

$$F\{u(x,t);k\} = \int_{-\infty}^{\infty} e^{ikx}[{}_x^{RF} D_{\theta}^{\alpha} u(x,t)]dx = -\psi_{\alpha}^{\theta}(k)\hat{u}(k,t), \quad k \in \mathbb{R}$$

$$\psi_{\alpha}^{\theta}(k) = |k|^{\alpha} e^{i(\text{sign}(k))\theta\frac{\pi}{2}} \tag{6}$$

We obtain

$$\left(\int_1^2 b(\beta)s^{\beta}d\beta\right)\hat{\tilde{u}}(k,s) - \left(\int_1^2 b(\beta)s^{\beta-1}d\beta\right)F(k) - \left(\int_1^2 b(\beta)s^{\beta-2}d\beta\right) G(k) + c^2\psi_{\alpha}^{\theta}(k)\hat{\tilde{u}}(k,s)$$
$$+ d^2\hat{\tilde{u}}(k,s) = \hat{\tilde{q}}(k,s)$$

From which

$$\hat{\tilde{u}}(k,s) = \frac{B(s) - \frac{d^2}{c^2}}{s(B(s)+\psi_{\alpha}^{\theta}(k))}F(k) + \frac{B(s) - \frac{d^2}{c^2}}{s^2(B(s)+\psi_{\alpha}^{\theta}(k))} G(k) + \frac{\hat{\tilde{q}}(k,s)}{c^2(B(s)+\psi_{\alpha}^{\theta}(k))} \tag{7}$$

Where $F(k),G(k)$ is the Fourier transform of the functions $f(x),g(x)$ respectively and

$$B(s) = \frac{1}{c^2}\left[\int_1^2 b(\beta)s^{\beta}d\beta + d^2\right]. \tag{8}$$

Now, by virtue of Bobylev-Cercignani theorem for inversion of the Laplace transform [18] of the functions $\hat{\tilde{u}}_1(k,s) = \frac{B(s)}{s^n(B(s)+\psi_{\alpha}^{\theta}(k))}$ and $\hat{\tilde{u}}_2(k,s) = \frac{1}{s^n(B(s)+\psi_{\alpha}^{\theta}(k))}$, we have the following

$$\hat{u}_j(k,t) = -\frac{1}{\pi}\int_0^{\infty} e^{-rt}\Im\{\hat{u}_j(k,re^{i\pi})\}dr. \quad j = 1,2 \tag{9}$$

In order to simplify the above relation (??), we need to evaluate the imaginary part of the functions $-\hat{u}_j(k,re^{i\pi})$ along the ray $s = re^{i\pi}$, $r > 0$ where the branch cut of the function s^{β} is defined. In this regard, by writing

$$B(re^{i\pi}) = \rho\cos\gamma\pi + i\rho\sin\gamma\pi, \quad \begin{cases} \rho = \rho(r) = \left|B(re^{i\pi})\right| \\ \gamma = \gamma(r) = \frac{1}{\pi}\arg[B(re^{i\pi})] \end{cases}$$

evaluating the imaginary part of the functions $-\hat{\tilde{u}}_j$

$$\Im\left\{\frac{B(s)}{s^n(B(s)+\psi_{\alpha}^{\theta}(k))}\right\} = K_1(n,k,r) = \frac{\psi_{\alpha}^{\theta}(k)\rho\sin(\pi\gamma)}{(-r)^n(\psi_{\alpha}^{\theta}(k)^2 + 2\psi_{\alpha}^{\theta}(k)\rho\cos(\pi\gamma) + \rho^2)} \quad n = 1,2$$

$$\Im\left\{\frac{1}{s^n(B(s)+\psi_{\alpha}^{\theta}(k))}\right\} = K_2(n,k,r) = \frac{-\rho\sin(\pi\gamma)}{(-r)^n(\psi_{\alpha}^{\theta}(k)^2 + 2\psi_{\alpha}^{\theta}(k)\rho\cos(\pi\gamma) + \rho^2)} \quad n = 0,1,2$$

and substituting in the relation (??), one leads to the following form

$$\hat{u}(k,t) = -\frac{F(k)}{\pi}\int_0^{\infty} e^{-rt}[K_1(1,k,r) - \frac{d^2}{c^2}K_2(1,k,r)]dr$$

$$-\frac{G(k)}{\pi}\int_0^{\infty} e^{-rt}[K_1(2,k,r) - \frac{d^2}{c^2}K_2(2,k,r)]dr$$

$$-\frac{1}{\pi c^2}\int_0^{\infty} e^{-rt}[\hat{q}(k,t) *_t K_2(0,k,r)]dr, \qquad (10)$$

Where $*_t$ is the convolution of the Laplace transform.

Also through the Fourier inversion of the function $\hat{u}(k,t)$, we can write the explicit solution $u(x,t)$ with respect to Green functions in the following form

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)G_1(x-\xi,t)d\xi - \int_{-\infty}^{\infty} g(\xi)G_2(x-\xi,t)d\xi -$$
$$\int_0^t q(\xi,\eta)G(x-\xi,t-\eta)d\eta,$$

Where the Green functions are denoted as

$$G_1(x,t) = -\frac{1}{2\pi x}\int_0^{\infty} e^{-rt}F^{-1}[K_1(1,k,r) - \frac{d^2}{c^2}K_2(1,k,r)]dr$$

$$G_2(x,t) = -\frac{1}{2\pi x}\int_0^{\infty} e^{-rt}F^{-1}[K_1(2,k,r) - \frac{d^2}{c^2}K_2(2,k,r)]dr$$

$$G(x,t) = -\frac{1}{\pi c^2}\int_0^{\infty} e^{-rt}F^{-1}[K_2(0,k,r)]dr d\xi, \qquad (11)$$

In special cases the Fourier inversions

$$F^{-1}[K(k);x] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}K(k)dk \qquad (12)$$

can be evaluated in terms of real and imaginary parts of the kernel $K(k)$

$$F^{-1}[K(k);x] = \frac{1}{\pi}\int_0^{\infty}\cos(kx)\Re(K(k))dk + \frac{1}{\pi}$$
$$\int_0^{\infty}\sin(kx)\Im(K(k))dk. \qquad (13)$$

The next section shows some simplifications of the relation (??).

The Symmetric Space-Time Fractional Klein-Gordon Equation of Distributed Order: For $\theta = 0$ we have a symmetric operator with respect to x, that can be interpreted as

$${}_x^{RF}D_0^{\alpha} = -(-\frac{d^2}{dx^2})^{\frac{\alpha}{2}}, \quad -|k|^{\alpha} = -(k^2)^{\frac{\alpha}{2}}$$

In this case we get *the symmetric space-time fractional Klein-Gordon equation of distributed order* which since the function $\hat{u}(k,t)$ is even in k, the inversion of the relation (??) takes the form

$$u(x,t) = \frac{1}{\pi}\int_0^{\infty}\cos(kx)\hat{u}^*(k,t)dk,$$

$$\hat{u}^*(k,t) = f(x) *_x \{-\frac{1}{\pi}\int_0^{\infty} e^{-rt}[K_1(1,k,r) - \frac{d^2}{c^2}K_2(1,k,r)]dr\}$$

$$-g(x) *_x \{\frac{1}{\pi}\int_0^{\infty} e^{-rt}[K_1(2,k,r) - \frac{d^2}{c^2}K_2(2,k,r)]dr\}$$

$$-q(x,t) *_t *_x \{\frac{1}{\pi c^2}\int_0^{\infty} e^{-rt}[K_2(0,k,r)]dr\}, \qquad (1)$$

Where $*_x$ is the convolution of the Fourier transform.

To calculate the Fourier integral we use the Mellin transform

$$M\{f(x);s\} = F(s) = \int_0^{\infty} x^{s-1}f(x)dx, \quad c_1 < \Re(s) < c_2$$

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{-s}ds, \quad c = \Re(s), \qquad (2)$$

With convolution theorem which implies that

$$M\{a(\xi) * b(\xi)\} = M\{\int_0^{\infty} a(\eta)b(\frac{\xi}{\eta})\frac{d\eta}{\eta}\} = A(s)B(s). \qquad (3)$$

By identifying the Fourier cosine integral in (??) as the Mellin convolution in k and setting

$$a(k,t) = \hat{u}^*(k,t), \quad b(k,x) = \frac{1}{\pi kx}\cos(\frac{1}{k}), \quad \xi = \frac{1}{x}, \eta = k,$$

the explicit solution $u(x,t)$ can be written as the Mellin inversion formula of product $A(s,t)B(s,x)$ namely

$$u(x,t) = \frac{1}{x}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} A(s,t)B(s,x)x^{-s}ds, \qquad (4)$$

Where $B(s,x)$ can be obtained from the Handbook by Erdelyi *et al.* [19, pp: 319] as follows

$$B(s,x) = \frac{\Gamma(1-s)}{x\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})} \quad 0 < \Re(s) < 1. \qquad (5)$$

For the required Mellin Transform of $A(s,t)$

$$A(s,t) = \int_0^{\infty}\hat{u}^*(k,t)k^{s-1}dk$$

Which depend on the terms in brackets in (??)

$$A(s,t)=\int_0^\infty \frac{e^{-rt}}{r}\{\frac{1}{\pi}\int_0^\infty[\frac{k^\alpha\rho\sin(\pi\gamma)}{(k^{2\alpha}+2k^\alpha\rho\cos(\pi\gamma)+\rho^2)}+\frac{d^2}{c^2}\frac{\rho\sin(\pi\gamma)}{(k^{2\alpha}+k^\alpha\rho\cos(\pi\gamma)+\rho^2)}]k^{s-1}dk\}dr$$

$$+\int_0^\infty \frac{e^{-rt}}{r^2}\{\frac{1}{\pi}\int_0^\infty[\frac{k^\alpha\rho\sin(\pi\gamma)}{(k^{2\alpha}+2k^\alpha\rho\cos(\pi\gamma)+\rho^2)}-\frac{d^2}{c^2}\frac{\rho\sin(\pi\gamma)}{(k^{2\alpha}+2k^\alpha\rho\cos(\pi\gamma)+\rho^2)}]k^{s-1}dk\}dr$$

$$+\frac{1}{\pi c^2}\int_0^\infty e^{-rt}\{\frac{1}{\pi}\int_0^\infty\frac{\rho\sin(\pi\gamma)}{(k^{2\alpha}+2k^\alpha\rho\cos(\pi\gamma)+\rho^2)}k^{s-1}dk\}dr. \quad (6)$$

We use change variable $k^\alpha = \rho y$ and apply the the following integral [19, pp: 309]

$$\frac{1}{\pi}\int_0^\infty\frac{\sin(\pi\gamma)}{y^2+2y\rho\cos(\pi\gamma)+1}y^{s-1}dy=-\frac{\sin((s-1)\pi\gamma)}{\sin(\pi s)}=-\frac{\pi\Gamma(s)\Gamma(1-s)}{\Gamma(\gamma(s-1))\Gamma(1-\gamma(s-1))}$$

$|\gamma| < 1, 0 < \Re(s) < 2$, (7)

to simplify the relation (??) into

$$A(s,t)=\int_0^\infty\frac{e^{-rt}}{r}\{-\frac{\rho^{\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2}}{\alpha}\frac{\Gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+3)\Gamma(1-(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+3))}{\Gamma(\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2))\Gamma(1-\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2))}$$

$$-\frac{d^2}{c^2}\frac{\rho^{\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+1}}{\alpha}\frac{\Gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2)\Gamma(1-(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2))}{\Gamma(\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+1))\Gamma(1-\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+1))}\}dr$$

$$+\int_0^\infty\frac{e^{-rt}}{r^2}\{-\frac{\rho^{\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2}}{\alpha}\frac{\Gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+3)\Gamma(1-(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+3))}{\Gamma(\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2))\Gamma(1-\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2))}$$

$$-\frac{d^2}{c^2}\frac{\rho^{\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+1}}{\alpha}\frac{\Gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2)\Gamma(1-(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2))}{\Gamma(\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+1))\Gamma(1-\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+1))}\}dr$$

$$-\int_0^\infty e^{-rt}\{\frac{\rho^{\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+1}}{\alpha}\frac{\Gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2)\Gamma(1-(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+2))}{\Gamma(\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+1))\Gamma(1-\gamma(\frac{s}{\alpha}-(\alpha+\frac{1}{\alpha})+1))}\}dr.$$

Finally, by substituting (??) in to(??) we can write the solution $u(x,t)$ with respect to Green functions in the following form

$$u(x,t)=\int_{-\infty}^\infty f(\xi)G_1(x-\xi,t)d\xi-\int_{-\infty}^\infty g(\xi)G_2(x-\xi,t)d\xi-\int_0^t q(\xi,\eta)G(x-\xi,t-\eta)d\eta,$$

Where the Green functions are denoted as

$$G_1(x,t)=-\frac{1}{\alpha\pi x}\int_0^\infty\frac{e^{-rt}}{r}[H(\frac{\sqrt[\alpha]{\rho}}{x})-\frac{d^2}{c^2}H^*(\frac{\sqrt[\alpha]{\rho}}{x})]dr$$

$$G_2(x,t)=\frac{1}{\alpha\pi x}\int_0^\infty\frac{e^{-rt}}{r^2}[H(\frac{\sqrt[\alpha]{\rho}}{x})+\frac{d^2}{c^2}H^*(\frac{\sqrt[\alpha]{\rho}}{x})]dr$$

$$G(x,t)=\frac{1}{\alpha\pi c^2}\int_{-\infty}^{\infty}\int_0^{\infty}e^{-rt}\frac{1}{x}H^*(\frac{\sqrt[\alpha]{\rho}}{x})drd\xi, \tag{8}$$

and the functions $H(\frac{\sqrt[\alpha]{\rho}}{x})$ and $H^*(\frac{\sqrt[\alpha]{\rho}}{x})$ are expressed in terms of Fox H-function shown in Appendix (B.1)

$$H(\frac{\sqrt[\alpha]{\rho}}{x})=\pi\rho^{2-(\alpha+\frac{1}{\alpha})}H_{3,4}^{2,1}\left[\frac{\sqrt[\alpha]{\rho}}{x}\left|\begin{matrix}(\alpha+\frac{1}{\alpha}-2,\frac{1}{\alpha});(\alpha+\frac{1}{\alpha}-2\gamma+1,\frac{\gamma}{\alpha});(1,\frac{1}{2})\\(\alpha+\frac{1}{\alpha}-2,\frac{1}{\alpha});(1,1);((\alpha+\frac{1}{\alpha}+2)\gamma+1,\frac{\gamma}{\alpha});(1,\frac{1}{2})\end{matrix}\right.\right]$$

$$H^*(\frac{\sqrt[\alpha]{\rho}}{x})=\pi\rho^{1-(\alpha+\frac{1}{\alpha})}H_{3,4}^{2,1}\left[\frac{\sqrt[\alpha]{\rho}}{x}\left|\begin{matrix}(\alpha+\frac{1}{\alpha}-1,\frac{1}{\alpha});(\alpha+\frac{1}{\alpha}-\gamma+1,\frac{\gamma}{\alpha});(1,\frac{1}{2})\\(\alpha+\frac{1}{\alpha}-1,\frac{1}{\alpha});(1,1);((\alpha+\frac{1}{\alpha}+1)\gamma+1,\frac{\gamma}{\alpha});(1,\frac{1}{2})\end{matrix}\right.\right],$$

provided that the integrals on the right-hand side of (??) are convergent.

The Space-Time Fractional Wave Equation of Single Order: In this case by setting $b(\beta)=\delta(\beta-n)$, $1<n<2$, $d=0$ the equation (??) is converted to space-time fractional wave equation of single order n,

$${}_t^C D_{0^+}^n u(x,t)-c^2[{}_x^{RF}D_\theta^\alpha]u(x,t)=q(x,t) \tag{1}$$

So that

$$B(s)=\frac{1}{c^2}s^n,\ \rho=\rho(r)=\frac{1}{c^2}r^n,\ \gamma=n.$$

Also, the transformed equation $\hat{\bar{u}}(k,s)$ (??), takes the form

$$\hat{\bar{u}}(k,s)=\frac{s^{n-1}}{s^n+\frac{1}{c^2}\psi_\alpha^\theta(k)}F(k)+\frac{s^{n-2}}{s^n+\frac{1}{c^2}\psi_\alpha^\theta(k)}G(k)+\frac{\hat{\bar{q}}(k,s)}{s^n+\frac{1}{c^2}\psi_\alpha^\theta(k)}. \tag{2}$$

Since, the inverse Laplace transform of $\frac{s^{n-m}}{s^n+\frac{1}{c^2}\psi_\alpha^\theta(k)}$ in (??) can be easily obtained as the Mittag-Leffler functions of order n [20]

$$L^{-1}\{\frac{s^{n-m}}{s^n+\frac{1}{c^2}\psi_\alpha^\theta(k)}\}=t^{m-1}E_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n)$$

the remaining solution with respect to Fourier inversion can be written as follows

$$u(x,t)=f(x)*_x\frac{1}{2\pi}\int_{-\infty}^{\infty}[E_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n)]e^{-ikx}dk$$

$$+g(x)*_x \frac{1}{2\pi}\int_{-\infty}^{\infty}[tE_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n)]e^{-ikx}dk$$

$$+q(x,t)*_x *_t \frac{1}{2\pi}\int_{-\infty}^{\infty}[t^n E_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n)]e^{-ikx}dk \tag{3}$$

Where $*_x, *_t$ is the convolutions of the Fourier and Laplace transforms respectively. To calculate the above integrals by writing Fourier kernel in real and imaginary part according to (??)

$$u(x,t) = f(x)*_x [\frac{1}{\pi}\int_0^{\infty}\cos(kx)\Re(E_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n))dk + \frac{1}{\pi}\int_0^{\infty}\sin(kx)\Im(E_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n))dk]$$

$$+g(x)*_x [\frac{t}{\pi}\int_0^{\infty}\cos(kx)\Re(E_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n))dk + \frac{t}{\pi}\int_0^{\infty}\sin(kx)\Im(E_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n))dk]$$

$$+q(x,t)*_x *_t [\frac{t^n}{\pi}\int_0^{\infty}\cos(kx)\Re(E_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n))dk + \frac{t^n}{\pi}\int_0^{\infty}\sin(kx)\Im(E_n(-\frac{1}{c^2}\psi_\alpha^\theta(k)t^n))dk]$$

and changing it in the Mellin convolution (??) similar to pervious section by knowing that [21]

$$b(k,x) = \frac{1}{\pi kx}\cos(\frac{1}{k}), \overset{M}{\rightarrow} B(s,x) = \frac{\Gamma(1-s)}{\pi x}\sin(\frac{\pi s}{2}) \quad 0 < \Re(s) < 1,$$

$$b(k,x) = \frac{1}{\pi kx}\sin(\frac{1}{k}), \overset{M}{\rightarrow} B(s,x) = \frac{\Gamma(1-s)}{\pi x}\cos(\frac{\pi s}{2}) \quad 0 < \Re(s) < 2,$$

$$a(k,t) = E_n(-\frac{t^n}{c^2}k^\alpha e^{\frac{i\pi\theta}{2}}), \overset{M}{\rightarrow} A(s,x) = \frac{1}{\alpha}\frac{\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-\frac{ns}{\alpha})}e^{\frac{-i\pi t^n\theta s}{2c^2\alpha}}$$

We can finally get the explicit solution $u(x,t)$ with respect to the Green function in terms of the Mellin-Barnes integral as follows

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)G(x-\xi,t)d\xi - t\int_{-\infty}^{\infty} g(\xi)G(x-\xi,t)d\xi - \int_0^t \eta^n q(\xi,\eta)G(x-\xi,t-\eta)d\eta,$$

Where the Green function is given by

$$G(x,t) = \frac{1}{\alpha\pi x}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})\Gamma(1-s)}{\Gamma(1-\frac{ns}{\alpha})}\sin[\frac{s\pi}{2\alpha}(\alpha-\frac{t^n\theta}{c^2})](\frac{1}{x})^s ds \tag{4}$$

$$G(x,t) = \frac{1}{\alpha\pi x}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})\Gamma(1-s)}{\Gamma(1-\frac{ns}{\alpha})}\sin[\frac{s\pi}{2\alpha}(\alpha-\frac{t^n\theta}{c^2})](\frac{1}{x})^s ds$$

$$= \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})\Gamma(1-s)}{\Gamma(\frac{\alpha-\frac{\theta t^n}{c^2}}{2\alpha}s)\Gamma(1-\frac{\alpha-\frac{\theta t^n}{c^2}}{2\alpha}s)\Gamma(1-\frac{ns}{\alpha})}(\frac{1}{x})^s ds \tag{5}$$

Also, depend on the variations of parameters α,β the reduced Green function can be shown as Fox H-function in the following form

$$G(x,t)=\frac{1}{\alpha x}H^{1,2}_{3,3}\left[\frac{1}{x}\left|\begin{matrix}(0,\frac{1}{\alpha});(0,1);(0,\frac{\alpha-\frac{\theta t^n}{c^2}}{2\alpha})\\(0,\frac{1}{\alpha});(0,\frac{\beta}{\alpha});(0,\frac{\alpha-\frac{\theta t^n}{c^2}}{2\alpha})\end{matrix}\right.\right],\quad \alpha<\beta$$

$$G(x,t)=\frac{1}{\alpha x}H^{1,2}_{3,3}\left[\frac{1}{x}\left|\begin{matrix}(1,\frac{1}{\alpha});(1,\frac{\beta}{\alpha});(1,\frac{\alpha-\frac{\theta t^n}{c^2}}{2\alpha})\\(1,\frac{1}{\alpha});(1,1);(1,\frac{\alpha-\frac{\theta t^n}{c^2}}{2\alpha})\end{matrix}\right.\right],\quad \alpha>\beta \tag{6}$$

When $1<\alpha=\beta<2$ the corresponding H-function is singular in $x=1$ and its singularity is removable. In this case representation of the Green function is written as the following elementary function [6]

$$G(x,t)=\frac{1}{\pi}\frac{x^{\alpha-1}\sin[\frac{\pi}{2}(\alpha-\frac{\theta t^n}{c^2})]}{1+2x^{\alpha}\cos[\frac{\pi}{2}(\alpha-\frac{\theta t^n}{c^2})]+x^{2\alpha}} \tag{7}$$

CONCLUSIONS

It may be concluded that on the basis of integral transform methods we developed analytical procedure for finding general solution of the linear space-time fractional Klein-Gordon equation of distributed order.

In special cases the Mellin transform is a supplementary tool to write the transformed equations in Mellin-Barens integrals and writing the Fox H-functions as the proper and well-suited functions in solutions of these equations.

Appendix A

For the well-behaved function $f(t)$, $t>0$ the so-called fractional derivative of order $\beta>0$ in the *Caputo* sense is defined as the operator ${}^{C}_{t}D^{\beta}_{0^+}$ such that

$${}^{C}_{t}D^{\beta}_{0^+}f(t)=\frac{1}{\Gamma(m-\beta)}\int_0^t\frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\beta+1-m}},\quad m-1<\beta<m \tag{A.1}$$

and in special case ${}^{C}_{t}D^{m}_{0^+}f(t)=\frac{d^m}{dt^m}f(t).$

The most important use of the Caputo fractional derivative is treated in initial-value problems where initial conditions are expressed in terms of integer-order derivatives. In this respect, it is interesting to know the Laplace transform of this type of derivative according to

$$L\{{}_t^C D_{0^+}^{\beta} f(t);s\} = s^{\beta}F(s) - \sum_{k=0}^{m-1} s^{\beta-1-k} f^{(k)}(0^+), \quad m-1 < \beta \le m \text{(A.2)}$$

Where

$$F(s) = L\{f(t);s\} = \int_0^{\infty} e^{-st} f(t)dt, \;\; s \in \mathbb{C}.$$

For the well-behaved function $f(x)$, $x \in \mathbb{R}$, the *Riesz-Feller* fractional derivative of order α and skewness θ is defined by

$${}_x^{RF}D_{\theta}^{\alpha} = \frac{\Gamma(1+\alpha)}{\pi}[\sin(\frac{\pi}{2}(\alpha+\theta))\int_0^{\infty}\frac{f(x+\xi)-f(\xi)}{\xi^{1+\alpha}}d\xi + \sin(\frac{\pi}{2}(\alpha-\theta))\int_0^{\infty}\frac{f(x-\xi)-f(\xi)}{\xi^{1+\alpha}}d\xi]\text{(A.3)}$$

and in secial case

$${}_x^{RF}D_{\pm 1}^{1} = \pm\frac{d}{dx}.$$

Also, for the Riesz-Feller fractional derivative of order it is interesting to know the Fourier transform of it namely

$$L\{{}_x^{RF}D_{\theta}^{\alpha} f(x);k\} = -|k|^{\alpha} e^{i(\text{sign}(k))\theta\frac{\pi}{2}} F(k), \quad k \in \mathbb{R}\text{(A.4)}$$

Where

$$F(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)dx$$

For further reading on the theory of fractional calculus, the interested reader is referred to [20, 21].

Appendix B

The Fox H-function is a generalized hypergeometric function defined by means of Mellin-Barens type contour integral as follows

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_1^p\\(b_k,B_k)_1^q\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \mathcal{H}_{p,q}^{m,n}(s)z^s ds \qquad z \ne 0\text{(B.1)}$$

Where the integrand $\mathcal{H}$ has the form in terms of the Gamma functions

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{A(s)B(s)}{C(s)D(s)}$$

$$A(s) = \prod_{k=1}^{m}\Gamma(b_k - B_k s), B(s) = \prod_{j=1}^{n}\Gamma\left(1-a_j+A_j s\right)$$

$$C(s) = \prod_{k=m+1}^{q}\Gamma(1-b_k+B_k s), D(s) = \prod_{j=n+1}^{p}\Gamma\left(a_j - A_j s\right)$$

and the orders (m,n,p,q) are non-negative integers such that $1 \le m \le q$, $0 \le n \le p$, the parameters $A_j > 0$, $B_k > 0$ are positive and α_j, b_k can be arbitrary complex such that the poles of the Gamma function entering the expressions $A(s)$, $B(s)$ are simple poles and do not coincide i.e.

$$A_j(b_k + l) \neq B_k(a_j - l' - 1) l, l' = 0,1,2,\cdots$$

$j = 1,\ldots, n$, $k = 1,\ldots, m$.

Remark: In the presence of a multiple pole s_0 of order n we need to expand the power series of the involved functions at the pole and evaluate the coefficient of the term $\frac{1}{s - s_0}$ as the residue. In this case the expansions of z^s and Gamma functions have the forms

$$z^s = z^{s_0}[1 + \log z(s - s_0) + O((s - s_0)^2)] \quad s \to s_0$$

$$\Gamma(s) = \Gamma(s_0)[1 + \psi(s_0)(s - s_0) + O((s - s_0)^2)] \quad s \to s_0, s_0 \neq 0,-1,-2,\ldots$$

$$\Gamma(s) = \frac{(-1)^k}{\Gamma(k+1)(s+k)}[1 + \psi(k+1)(s+k) + O((s+k)^2)], \quad s \to -k, k = 0,1,2,\ldots$$

Where $\psi(z)$ is the logarithmic derivative of the Gamma function $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Also, the contour L can be chosen as follows

$L = (\gamma - i\infty, \gamma + i\infty), \gamma \in \mathbb{R}$ is a contour starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty$ and leaving to the right all the poles of $A(s)$ and to the left all the poles of $B(s)$.

$L_{+\infty}$ is a loop beginning and ending at $+\infty$ and encircling once in the negative direction all the poles of $A(s)$, but none of the poles of $B(s)$.

$L_{-\infty}$ is a loop beginning and ending at $-\infty$ and encircling once in the negative direction all the poles of $B(s)$, but none of the poles of $A(s)$.

Furthermore, depend on the following parameters

$$\rho = \prod_{j=1}^{p} A_j^{-A_j} \prod_{k=1}^{q} B_k^{B_k}, \Delta = \sum_{k=1}^{q} B_k - \sum_{j=1}^{p} A_j$$

$$\mu = \sum_{k=1}^{q} b_k - \sum_{j=1}^{p} a_j + \frac{p-q}{2} a^* = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{k=1}^{m} B_k - \sum_{k=m+1}^{q} B_k$$

the choices of the contour L and convergence domains for analytic function H can be found. For more details about this functions such as convergency, analytic continuation and their application in applied sciences the reader is referred to [22-26].

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