

Some Characterizations of Chaudhry and Zubair (2002)'S Generalization of the Generalized Inverse Gaussian Distribution by Truncated Moments

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Abstract: It is important to characterize a probability distribution subject to certain conditions before it is applied to real world data. The objective of this paper is to characterize the Chaudhry and Zubair (2002) 's generalization of the generalized inverse Gaussian. We have established several new characterization results of the Chaudhry and Zubair (2002) 's generalization of the generalized inverse Gaussian distribution. Our proposed characterizations are based on the left and right truncated moments, order statistics and upper record values.

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INTRODUCTION

For probability distributions, the characterization problems have been studied by many authors and researchers, see, for example, Ahsanullah [2], Ahsanullah *et al.* [3, 4, 5], Ahsanullah *et al.* [4], Galambos and Kotz [6], Glänzel [7], Kotz and Shanbhag [8] and Nagaraja [9] and references therein. It is important to characterize a probability distribution subject to certain conditions before it is applied to real world data; see, for example, Nagaraja [9] and Koudou and Ley [10], among others. A characterization of a particular probability distribution states that it is the only distribution that satisfies some specified conditions. According to Glänzel [11], these characterizations may serve as a basis for parameter estimation. Further, Glänzel [11] points out that the characterizations by truncated moments may also be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions. Also, see Glänzel [7]. These conditions are used by various authors to test goodness of fit, efficiency of a particular test of hypothesis and the power of a particular estimating, etc. For example, Volkova and Nikitin [12] used a well-known characterization result of Ahsanullah [13] to test exponentiality of a distribution. For details, see

Volkova and Nikitin [12]. For an excellent survey of goodness-of-fit and symmetry tests based on the characterization properties of distributions, the interested readers are referred to two recent nice papers of Nikitin [14] and Milošević [15], respectively and references therein. Many authors and researchers have studied the characterizations of the generalized inverse Gaussian (GIG) distribution; see, for example, Chou and Huang [16], among others. For a very nice and detailed survey on the characterizations of GIGD, the interested readers are also referred to Koudou and Ley [10].

Chaudhry and Zubair [1] introduced a generalization of the generalized inverse Gaussian distribution (GGIGD). Recently, Shakil and Ahsanullah [17] studied the Chaudhry and Zubair [1]'s generalization of the generalized inverse Gaussian distribution, including the reliability analysis, the estimation of the parameters, computations of percentage points, applications to some real life-time data and drew some inferences on it. The generalized inverse Gaussian (GIG) distribution has received special attention in view of its wide applications in many areas of research such as actuaries, biomedicine, demography, environmental and ecological sciences, finance, lifetime data, reliability theory and traffic data, among others, since it was first proposed by Halphen in 1946; see, for example, Perrault *et al.* [18], Seshadri [19],

Jorgensen [20], Chhikara and Folks [21], Johnson *et al.* [22], Marshall and Olkin [23] and Seshadri [24] and references therein.

It appears from the literature that no attention has been paid to the characterizations of the Chaudhry and Zubair [1]'s generalization of the generalized inverse Gaussian distribution (GGIGD) since it appeared in their seminal work viz.: Chaudhry and Zubair [1]. Motivated by the importance of characterizations of probability distributions in applied research, in this paper, we have established several new characterization results of Chaudhry and Zubair [1]'s GGIGD by the left and right truncated moments, order statistics and upper record values. It is hoped that these characterizations may serve as a basis for parameter estimation and in developing some goodness-of-fit tests of the Chaudhry and Zubair [1]'s GGIGD. Moreover, we believe that the findings of this paper will be quite useful for the researchers and practitioners in various fields of theoretical and applied sciences.

The organization of the paper is as follows: For the sake of completeness and interest of the readers, we have provided some preliminaries, including a brief description of Chaudhry and Zubair [1]'s generalization of the generalized inverse Gaussian distribution (GGIGD). We have established our proposed new characterization results of Chaudhry and Zubair [1]'s GGIGD by left and right truncated

moments, order statistics and upper record values. Some concluding remarks are provided.

Some Preliminaries: In what follows, for the sake of completeness and interest of the readers, following Shakil and Ahsanullah [17], we will briefly describe the Chaudhry and Zubair [1]'s GGIGD and some of its essential distributional properties which will be useful in establishing our proposed characterization results. For a continuous positive random variable X , Chaudhry and Zubair [1] introduced a generalization of the generalized inverse Gaussian distribution with its probability density function (pdf) given by;

$$f_X(x) = C x^{\alpha-1} \exp(-ax) W_{0, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right), \quad (1)$$

where $x > 0, \nu \geq 0, a > 0, b \geq 0, -\infty < \alpha < \infty$ and

$$C = C(\alpha; a, b, \nu) = \left(\int_0^\infty x^{\alpha-1} \exp(-ax) W_{0, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right) dx \right)^{-1} \quad (2)$$

Denotes the normalizing constant and $W_{\kappa, \nu}(z)$ denotes the Whittaker function for reals κ and ν and real argument z ; see, for example, Lebedev [25]. Using the Eq. 4.14, P. 197 of Chaudhry and Zubair [1], the normalizing constant is easily given by the following formula.

$$C(\alpha; a, b, \nu) = \left(\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha-2} \pi^{-1} (ab)^{\frac{1}{2}} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right) \right)^{-1} \quad (3)$$

where $G_{0,4}^{4,0}(\cdot)$ denotes the Meijer G-function; see, for example, Mathai [26]. Furthermore, it is known that the Whittaker function, $W_{\lambda, \mu}(z)$, is related to the confluent hypergeometric function $\Psi([], []; z)$ by the following formula.

$$W_{\lambda, \mu}(z) = z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}} \Psi\left(\frac{1}{2} - \lambda + \mu, 2\mu + 1, z\right), \arg|z| < \pi$$

Lebedev ([25], Eq. 9.13.16, p. 274), hence, using the above-mentioned formula, the equations (1) and (2) can easily be transformed in terms of the Whittaker function. Thus, we have,

$$f_X(x) = C (2b)^{\nu+1} x^{\alpha-\nu-2} e^{-\left(ax + \frac{b}{x}\right)} \Psi\left(\nu + 1, 2(\nu + 1); \frac{2b}{x}\right), \quad (4)$$

where,

$$C = C(\alpha; a, b, \nu) = \left((2b)^{\nu+1} \int_0^\infty x^{\alpha-\nu-2} e^{-\left(ax + \frac{b}{x}\right)} \Psi\left(\nu + 1, 2(\nu + 1); \frac{2b}{x}\right) dx \right)^{-1}, \quad (5)$$

where $x > 0, v \geq 0, a > 0, b \geq 0, -\infty < \alpha < \infty$.

Also, since it is known from Gradshteyn and Ryzhik ([27], Eq. 9.235.2, p. 1062) that the Whittaker function, $W_{0,\mu}(z)$, is related to the Macdonald function $K_\mu(z)$ by the following formulas.

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} W_{0,\mu}(2z),$$

that is,

$$W_{0,\mu}(2z) = \left(\frac{2z}{\pi}\right)^{1/2} K_\mu(z),$$

Hence, in view of above-mentioned formulas, the equations (1) and (2) can also be easily simplified in terms of the Macdonald function. Thus, we have

$$f_X(x) = C \left(\frac{2b}{\pi}\right)^{1/2} x^{\alpha-3/2} e^{-ax} K_{v+1/2}\left(\frac{b}{x}\right), \quad (6)$$

where

$$C = C(\alpha, a, b, v) = \left[\left(\frac{2b}{\pi}\right)^{1/2} \int_0^\infty \left(\frac{2b}{\pi}\right)^{1/2} x^{\alpha-3/2} e^{-ax} K_{v+1/2}\left(\frac{b}{x}\right) dx \right]^{-1}, \quad (7)$$

where $x > 0, v \geq 0, a > 0, b \geq 0, -\infty < \alpha < \infty$.

The cumulative distribution function (cdf) corresponding to the pdf (1) is given by;

$$F_X(x) = C \int_0^x t^{\alpha-1} \exp(-at) W_{0,v+1/2}\left(\frac{2b}{t}\right) dt, \quad (8)$$

which obviously cannot be integrated analytically in closed form and so should be evaluated numerically.

Furthermore, since it is known from Gradshteyn and Ryzhik ([27], Eq. 9.235.2, P. 1062) that $W_{0,v}(2z) = \left(\frac{2z}{\pi}\right)^{1/2} K_v(z)$, hence, in view of this formula, the equations (8) can be simplified in terms of the Macdonald

function $K_v(z)$. Chaudhry and Zubair ([28] and [1]) introduced the following generalizations of the generalized incomplete gamma functions:

$$\gamma_v(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_0^x t^{\alpha-3/2} \exp(-t) K_{v+1/2}\left(\frac{b}{t}\right) dt, \quad (9)$$

and

$$\Gamma_v(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_x^\infty t^{\alpha-3/2} \exp(-t) K_{v+1/2}\left(\frac{b}{t}\right) dt, \quad (10)$$

where α, x are complex parameters, b is a complex variable and $K_v(z)$ denotes the modified Bessel function of the second kind or the Macdonald function for complex parameter v and complex argument z . Note that for v real and z positive, $K_v(z)$ is real. For details on the theory and analytical properties of Bessel functions, the interested readers are referred to Watson [29]. Thus, by using the definitions (18), the cumulative distribution function (cdf), $F_X(x)$, corresponding to the pdf (1), is given in terms of the generalizations of the generalized incomplete gamma functions by;

$$F_X(x) = C a^{-\alpha} \gamma_v(\alpha, ax; ab), \quad (11)$$

The moments of the Chaudhry and Zubair [1]'s GGIG distribution are given as follows:

For positive integer k , the k th moment of the random variable X of the Chaudhry and Zubair [1]'s GGIGD is given by

$$E(X^k) = (C) \int_0^\infty x^{\alpha+k-1} \exp(-ax) W_{0, \nu + \frac{1}{2}} \left(\frac{2b}{x} \right) dx, \quad (12)$$

where $x > 0$, $\nu \geq 0$, $a > 0$, $b \geq 0$, $-\infty < \alpha < \infty$ and

$$C = \left[\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha-2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right) \right]^{-1}. \quad (13)$$

Using the Eq. 4.14, P. 197 of Chaudhry and Zubair [1], the k th moment is easily given by the following formula:

$$E(X^k) = (C) \frac{1}{a^{\alpha+k-\frac{1}{2}}} 2^{\alpha+k-2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + k + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + k - \frac{1}{2} \right) \right. \right), \quad (14)$$

where C is given as mentioned above and $G_{0,4}^{4,0}(\cdot)$ denotes the Meijer G-function; see, for example, Mathai [26]. Taking $k = 1$, in the above-mentioned equation of the k th moment, the mean (or the first moment) of the random variable X is easily given by;

$$E(X) = (C) \frac{1}{a^{\alpha+\frac{1}{2}}} 2^{\alpha-1} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{3}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right) \right. \right) \quad (15)$$

The first incomplete moment of the random variable X is given as;

$$I_X(x) = \int_0^x u f(u) du = C \int_0^x u \exp(-au) W_{0, \nu + \frac{1}{2}} \left(\frac{2b}{u} \right) du = C \gamma_\nu(2, ax; ab), \quad (16)$$

Which follows from Eq. (9) and the formula $W_{0,\nu}(2z) = \left(\frac{2z}{\pi}\right)^{1/2} K_\nu(z)$.

Characterization Results: In what follows, in this section, we establish some new characterization results of the distribution of Chaudhry and Zubair's generalization of the generalized inverse Gaussian distribution. Our proposed characterizations are based on the left and right truncated moments, order statistics and upper record values. For this, we will need the following assumption and lemmas.

Assumption 1: Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = \inf \{x \mid F(x) > 0\}$, and $\delta = \sup \{x \mid F(x) < 1\}$. We also assume that $f(x)$ is a differentiable function for all x and $E(X)$ exists.

Lemma 1: Under the Assumption 1, if $E(X|X \leq x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable

function of x with the condition that $\int_0^x \frac{u - g'(u)}{g(u)} du$ is finite for $x > 0$, then $f(x) = c e^{\int_0^x \frac{u - g'(u)}{g(u)} du}$, where c is a constant determined by the condition $\int_0^\infty f(x) dx = 1$.

Proof: The proof of Lemma 1 can be found in Ahsanullah *et al.* [4], or Shakil *et al.* [30].

Lemma 2: Under the Assumption 1, if $E(X | X \geq x) = \tilde{g}(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $\tilde{g}(x)$ is a continuous

differentiable function of x with the condition that $\int_x^\infty \frac{u + \left[\tilde{g}(u) \right]'}{\tilde{g}(u)} du$ is finite for $x > 0$, then $f(x) = c e^{-\int_0^x \frac{u + \left[\tilde{g}(u) \right]'}{\tilde{g}(u)} du}$,

where c is a constant determined by the condition $\int_0^\infty f(x) dx = 1$.

Proof: The proof of Lemma 2 can be found in Ahsanullah *et al.* [4], or Shakil *et al.* [30].

Characterization by Truncated Moment: In what follows, we will establish our proposed characterization results by the left and right truncated moments.

Theorem 1: If the random variable X satisfies the Assumption 1 with $\omega = 0$ and $\delta = \infty$, then $E(X | X \leq x) = g(x) \frac{f(x)}{F(x)}$, where,

$$g(x) = \frac{\gamma_v(2, ax; ab)}{\exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)}, \quad (17)$$

if and only if X has the pdf $f(x) = C(\alpha; a, b) \exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)$, where,

$$C(\alpha; a, b) = \left(\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha - 2} \pi^{-1} (ab)^{1/2} G_{0, 4}^{4, 0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(v + \frac{1}{2} \right), \frac{-1}{2} \left(v + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right) \right)^{-1}.$$

Proof: Suppose that $E(X | X \leq x) = g(x) \frac{f(x)}{F(x)}$. Then, since $E(X | X \leq x) = \frac{\int_0^x u f(u) du}{F(x)}$, we have $g(x) = \frac{\int_0^x u f(u) du}{f(x)}$,

where the numerator, $\int_0^x u f(u) du$, denotes the incomplete moment given by (16). Now, if the random variable X satisfies the Assumption 1 and has the distribution with the pdf (1), then we have;

$$g(x) = \frac{\int_0^x u f(u) du}{f(x)} = \frac{\int_0^x u \exp(-au) W_{0, v + \frac{1}{2}}\left(\frac{2b}{u}\right) du}{\exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)} = \frac{\gamma_v(2, ax; ab)}{\exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)}$$

which follows by using the expressions for the pdf (1) and the incomplete moment (16).

Conversely, suppose that

$$g(x) = \frac{\gamma_v(2, ax; ab)}{\exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)}.$$

Since, by Lemma 1, $g'(x) = x - g(x) \frac{f'(x)}{f(x)}$, (see Ahsanullah *et al.* [4], or Shakil *et al.* [30]), differentiating $g(x)$ with respect to x , we have

$$g'(x) = x - g(x) \left(-a + \frac{d}{dx} \ln \left(W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right) \right) \right),$$

from which, using Eq. 9.234.3, P. 1062, of Gradshteyn and Ryzhik [27], we obtain,

$$\frac{x - g'(x)}{g(x)} = -a + \frac{d}{dx} \ln \left(W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right) \right) = -a - \frac{b}{x^2} + \frac{1}{x} \frac{W_{1, v + \frac{1}{2}}\left(\frac{2b}{x}\right)}{W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)}.$$

Now, since, by Lemma 1, we have,

$$\frac{x - g'(x)}{g(x)} = \frac{f'(x)}{f(x)},$$

it follows that

$$\frac{f'(x)}{f(x)} = -a - \frac{b}{x^2} + \frac{1}{x} \frac{W_{1, v + \frac{1}{2}}\left(\frac{2b}{x}\right)}{W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)}.$$

On integrating the above-mentioned equation with respect to x and simplifying, we obtain.

$$\ln f(x) = \ln \left(c \exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right) \right),$$

or,

$$f(x) = c \exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right),$$

where c is the normalizing constant to be determined. Thus, on integrating the above-mentioned equation with respect to x from $x = 0$ to $x = \infty$ and using the condition $\int_0^\infty f(x) dx = 1$, we obtain.

$$c = \left(\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha - 2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \middle| \frac{1}{2} \left(v + \frac{1}{2} \right), \frac{-1}{2} \left(v + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right) \right)^{-1}.$$

This completes the proof of Theorem 1.

Theorem 2: If the random variable X satisfies the Assumption 1 with $\omega = 0$ and $\delta = 0$, then $E(X|X \geq x) = \tilde{g}(x) \frac{f(x)}{1-F(x)}$, where,

$$\tilde{g}(x) = \left(\frac{(E(X) - g(x)f(x))}{\exp(-ax)W_{0, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right)} \right) \times \frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha-2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right),$$

where $g(x)$ is given by Eq. (17) and $E(X)$ is given by Eq. (15), if and only if X has the pdf $f(x) = C(\alpha; a, b) \exp(-ax) W_{0, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right)$, where,

$$C(\alpha; a, b) = \left(\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha-2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right) \right)^{-1}.$$

Proof: Suppose that $E(X|X \geq x) = \tilde{g}(x) \frac{f(x)}{1-F(x)}$. Then, since $E(X|X \geq x) = \frac{\int_x^\infty u f(u) du}{1-F(x)}$, we have $\tilde{g}(x) = \frac{\int_x^\infty u f(u) du}{f(x)}$.

Now, if the random variable X satisfies the Assumption 1 and has the distribution with the pdf (1), then we have.

$$\begin{aligned} \tilde{g}(x) &= \frac{\int_x^\infty u f(u) du}{f(x)} = \frac{\int_0^\infty u f(u) du - \int_0^x u f(u) du}{f(x)} \\ &= \left(\frac{(E(X) - g(x)f(x))}{\exp(-ax)W_{0, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right)} \right) \\ &\quad \times \frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha-2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right) \end{aligned}$$

Conversely, suppose that $\tilde{g}(x) = \left(\frac{(E(X) - g(x)f(x))}{\exp(-ax)W_{0, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right)} \right) \times \frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha-2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right).$

By Lemma 2, since $\left(\tilde{g}(x) \right)' = -x - \tilde{g}(x) \frac{f'(x)}{f(x)}$, (for details, see Ahsanullah *et al.* [4], Shakil *et al.* [30]), therefore differentiating $\tilde{g}(x)$ with respect to x , we have.

$$\left(\tilde{g}(x)\right)' = -x - \tilde{g}(x) \left(-a + \frac{d}{dx} \ln \left(W_{0, \nu + \frac{1}{2}} \left(\frac{2b}{x} \right) \right) \right),$$

from which we obtain

$$\frac{x + \left(\tilde{g}(x)\right)'}{\tilde{g}(x)} = - \left(-a + \frac{d}{dx} \ln \left(W_{0, \nu + \frac{1}{2}} \left(\frac{2b}{x} \right) \right) \right) = -a - \frac{b}{x^2} + \frac{1}{x} \frac{W_{1, \nu + \frac{1}{2}} \left(\frac{2b}{x} \right)}{W_{0, \nu + \frac{1}{2}} \left(\frac{2b}{x} \right)}.$$

Now, since, by Lemma 2, we have

$$\frac{f'(x)}{f(x)} = - \frac{x + \left[\tilde{g}(x)\right]'}{\tilde{g}(x)},$$

it follows that

$$\frac{f'(x)}{f(x)} = -a - \frac{b}{x^2} + \frac{1}{x} \frac{W_{1, \nu + \frac{1}{2}} \left(\frac{2b}{x} \right)}{W_{0, \nu + \frac{1}{2}} \left(\frac{2b}{x} \right)}.$$

On integrating the above-mentioned equation with respect to x and simplifying, we obtain.

$$\ln f(x) = \ln \left(c \exp \left(-ax \right) W_{0, \nu + \frac{1}{2}} \left(\frac{2b}{x} \right) \right),$$

or,

$$f(x) = c \exp \left(-ax \right) W_{0, \nu + \frac{1}{2}} \left(\frac{2b}{x} \right),$$

where c is the normalizing constant to be determined. Thus, on integrating the above-mentioned equation with respect to x from $x = 0$ to $x = \infty$ and using the condition $\int_0^\infty f(x) dx = 1$, we obtain.

$$c = \left(\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha - 2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \middle| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right) \right)^{-1}.$$

This completes the proof of Theorem 2.

Characterizations by Order Statistics: If X_1, X_2, \dots, X_n be the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and pdf $f(x)$ and if $X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n}$ be the corresponding order statistics, then it is known from Ahsanullah *et al.* [31], chapter 5, or Arnold *et al.* [32], chapter 2, that $X_{j,n} | X_{k,n} = x$, for $1 \leq k < j \leq n$, is distributed as the $(j - k)$ th order statistics from $(n - k)$ independent observations from the random variable V having the pdf $f_V(v|x)$ where $f_V(v|x) = \frac{f(v)}{1 - F(x)}$, $0 \leq v < x$ and $X_{i,n} | X_{k,n} = x$, $1 \leq i < k \leq n$, is distributed as i th order

statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where $f_W(w|x) = \frac{f(w)}{F(x)}$, $w < x$. Let $S_{k-1} = \frac{1}{k-1} (X_{1,n} + X_{2,n} + \dots + X_{k-1,n})$ and $T_{k,n} = \frac{1}{n-k} (X_{k+1,n} + X_{k+2,n} + \dots + X_{n,n})$.

Theorem 3: Suppose the random variable X satisfies the Assumption 1 with $\omega = 0$ and $\delta = 0$, then $E(S_{k-1} | X_{k,n} = x) = g(x) \tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and.

$$g(x) = \frac{\gamma_v(2, ax; ab)}{\exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)},$$

if and only if X has the pdf $f(x) = C(\alpha; a, b) \exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)$, where,

$$C(\alpha; a, b) = \left(\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha - 2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(v + \frac{1}{2} \right), \frac{-1}{2} \left(v + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right) \right)^{-1}.$$

Proof: It is known that $E(S_{k-1} | X_{k,n} = x) = E(X | X \leq x)$; see Ahsanullah *et al.* [31] and David and Nagaraja [32]. Hence, by Theorem 1, the result follows.

Theorem 4: Suppose the random variable X satisfies the Assumption 1 with $\omega = 0$ and $\delta = 0$, then $E(T_{k,n} | X_{k,n} = x) = g(x) \frac{f(x)}{1 - F(x)}$, where

$$\begin{aligned} \tilde{g}(x) &= \left(\frac{(E(X) - g(x)f(x))}{\exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)} \right) \\ &\times \frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha - 2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(v + \frac{1}{2} \right), \frac{-1}{2} \left(v + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right), \end{aligned}$$

where $g(x)$ is given by Eq. (17) and $E(X)$ is given by Eq. (15), if and only if X has the pdf $f(x) = C(\alpha; a, b) \exp(-ax) W_{0, v + \frac{1}{2}}\left(\frac{2b}{x}\right)$, where

$$C(\alpha; a, b) = \left(\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha - 2} \pi^{-1} (ab)^{1/2} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(v + \frac{1}{2} \right), \frac{-1}{2} \left(v + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right) \right)^{-1}.$$

Proof: Since $E(T_{k,n} | X_{k,n} = x) = E(X | X \geq x)$, see Ahsanullah *et al.* [31] and David and Nagaraja [32], the result follows from Theorem 2.

Characterization by Upper Record Values: For details on record values, see Ahsanullah [33]. Let X_1, X_2, \dots be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. If $Y_n = \max(X_1, X_2, \dots, X_n)$ for $n \leq 1$ and $Y_j > Y_{j-1}$, $j > 1$, then X_j is called an upper record value of $\{X_n, n \geq 1\}$. The indices at which the upper records occur are given by the record times $\{U(n) > \min(j | j > U(n+1), X_j > X_{U(n-1)}, n > 1)\}$ and $U(1) = 1$. Let the n th upper record value be denoted by $X_{(n)} = X_{U(n)}$.

Theorem 5: Suppose the random variable X satisfies the Assumption 1 with $\omega = 0$ and $\delta = 0$, then $E(X(n+1) | X(n) = x) = \tilde{g}(x) \frac{f(x)}{1 - F(x)}$, where

$$\tilde{g}(x) = \left(\frac{(E(X) - g(x)f(x))}{\exp(-ax)W_{0, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right)} \right) \\ \times \frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha - 2} \pi^{-1} (ab)^{\frac{1}{2}} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right),$$

where $g(x)$ is given by Eq. (17) and $E(X)$ is given by Eq. (15), if and only if X has the pdf $f(x) = C(\alpha; a, b) \exp(-ax) W_{0, \nu + \frac{1}{2}}\left(\frac{2b}{x}\right)$, where

$$C(\alpha; a, b) = \left(\frac{1}{a^{\alpha - \frac{1}{2}}} 2^{\alpha - 2} \pi^{-1} (ab)^{\frac{1}{2}} G_{0,4}^{4,0} \left(\frac{(ab)^2}{16} \left| \frac{1}{2} \left(\nu + \frac{1}{2} \right), \frac{-1}{2} \left(\nu + \frac{1}{2} \right), \frac{1}{2} \left(\alpha + \frac{1}{2} \right), \frac{1}{2} \left(\alpha - \frac{1}{2} \right) \right. \right) \right)^{-1}.$$

Proof: It is known from Ahsanullah *et al.* [31] and Nevzorov [34], that $E(X(n+1) | X(n) = x) = E(X | X \geq x)$. Then, the result follows from Theorem 2.

Concluding Remarks: It is important to characterize a probability distribution subject to certain conditions before it is applied to real world data. Motivated by the importance of characterizations of probability distributions in applied research, in this paper, we have established some new characterization results by truncated moment, order statistics and upper record values of Chaudhry and Zubair (2002)'s generalization of the generalized inverse Gaussian. It is hoped that the findings of this paper will be quite useful to the researchers and practitioners in various fields of theoretical and applied sciences, such as biomedicine, demography, environmental science, ecological science, finance, lifetime and quantum plasmadynamics, among others.

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