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Numerical Study for Solving Quadratic Riccati Differential Equations

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Abstract: There has been greater attempt to solving differential equations by numerical methods. Most of authors treated numerical approach to solve nonlinear Riccati differential equation (RDE). Numerical Laplace transform method is applied to approximate the solution of nonlinear quadratic Riccati differential equations mingled with Adomian decomposition method. A new technique is given by Vinod M and Dimple R by reintroducing the unknown function in Adomian polynomial with that of well-known Newton-Raphson formula. Recent. N IDE studied this problem to solve nonlinear Riccati differential equation by numerical method using of Newton's interpolation and Aitken's method as a hybrid technique by using these two types of interpolation, some examples in which comparisons are made among the Numerical Laplace transform method, exact solutions, ADM (Adomian decomposition method), HPM (Homotopy perturbation method), Taylor series method and the method Proposed by Vinod M and Dimple R, we compared also the result for some examples with exact solution, variational iteration method (VIM) and multistage variational method (MVIM). We found that using this method is not generally good, as for (VIM) and (MVIM) methods and this method can be improved by taking a small step h for the solution interval and obtaining the approximation relationship, then using it in a limited number of first points of the solution interval.

Key words: Quadratic Riccati differential equation · Numerical method · Newton's interpolation and Aitken's method · Variational iteration method · Multistage variational iteration method

INTRODUCTION

Conceder the nonlinear Riccati differential equation (RDE) of the form [1]:

$$\frac{dy}{dx} = q(x)y + r(x)\,y^2 + p(x),\ \ y(0) = a \tag{1}$$

where q(x), r(x) and p(x) are the known scalar functions and a is an arbitrary constant. This equation named after the name of Italian nobleman Count Jacopo Francesco Riccati(1676-1754) [1-4]. The applications of this equation may be found in some kinds of applied sciences. In [1-3], N IDE applied numerical method to solve the RDE, In [5] Ghorbani and Momani applied the piecewise variational iteration method (VIM) to solve the RDE. Differential transform method [6] is adopted to find the solution of RDE. Taiwo and Osilagun [7] approximated the solution of RDE by Iterative algorithm. Perturbation iteration algorithm (PIA) has been presented in solving RDE [8]. Vahidi has made the comparison among HPM, ADM and LTDM in solving RDE in [9]. In [10] the authors developed the iterative methods ADM, MADM, VIM, MVIM, HPM, MHPM and HAM to solve the general RDE. Laplace transform is a powerful tool in solving linear problems but it is incapable of solving nonlinear problems. A well-known numerical algorithm Laplace transforms and Adomian decomposition method has conquered much importance in solving many linear and nonlinear problems which provides a series solution. Suheil A. Khuri was the first to apply Laplace decomposition algorithm to solve a class of nonlinear differential equation [11]. A combined Laplace Adomian decomposition method is used to solve nonlinear Volterra integral equation with weakly kernel [12]. In [13], Majid Khan et. al. solved nonlinear coupled partial differential equations with the help of Laplace Decomposition method. LDM is also implemented to obtain the series solution of nonlinear fractional differential equations [14]. Waleed Al-Hyani [15] solved nth order Integro differential equations by the usage of LT-ADM. In [16], Modified Laplace decomposition method is proposed for solving

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Lane-Emden type differential equation. LDM is exercised to solve the Logistic differential equations in [17]. Wazwaz [18] employed CLT-ADM for solving nonlinear volterra-integro differential equations. For handling the solutions of nonlinear system of partial differential equation Laplace decomposition method and pade approximant is used in [19]. Hence this method is utilized to solve many more problems like Singular initial value problems [20], Double singular boundary value problems [21], Higher order boundary value problems [22]. Other cited references are [23-29]. In Vinod M and Dimple R. [1] uses the Laplace transform-Adomian decomposition method to solve the Quadratic RDE. They replace the unknown function y_i in Adomian polynomial with Newton-Raphson formula, which improves the Adomian polynomial. Faith Chelimo Kosgei [30] studied the problem of solution of first order differential equation using numerical Newton's interpolation and Lagrange method by combined the newton's interpolation and Lagrange method, In this study we will combine of Newton's interpolation and Aitken's method instead of Lagrange method to solve RDE, [31-34]. Finally we verified on a number of examples and numerical results obtained show the efficiency of the method given by present study in comparison with Vinod M, Dimple R [4] and with Belal Batiha [35].

Combined newton's interpolation and Lagrange Method [30-34]: This study combine Newton's interpolation method and Lagrange method it used newton's interpolation method to find the second two terms then use the three values for y to form a quadratic equation using Lagrange interpolation method as follows;

Newton's Interpolation Method:

$$f_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + ... + a_n(x-x_0)(x-x_1)...a_2(x-x_{n-1}) \tag{2}$$

where

$$a_0 = y_0,\ a_1 = \frac{f(x_1)-f(x_0)}{(x_1-x_0)},\ a_2 = \frac{\frac{f(x_2-x_1)}{(x_2-x_1)} - \frac{f(x_1-x_0)}{(x_1-x_0)}}{(x_2-x_0)} \tag{3}$$

Etc

Lagrang Interpolation Method:

$$y_n = \frac{(x-x_1)-(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)-(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)-(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2 \tag{4}$$

Description of the Proposed Method: This method will combine a Newton's interpolation method and Aitken method. It used newton's interpolation method to find the second two terms then use the three values for y to form a linear or quadratic equations using Aitken interpolation method as follows;

$$f_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + ... + a_n(x-x_0)(x-x_1)...a_2(x-x_{n-1}) \tag{5}$$

where

$$a_0 = y_0,\ a_1 = \frac{f(x_1)-f(x_0)}{(x_1-x_0)},\ a_2 = \frac{\frac{f(x_2-x_1)}{(x_2-x_1)} - \frac{f(x_1-x_0)}{(x_1-x_0)}}{(x_2-x_0)} \tag{6}$$

etc

$$y_1 = a_0 + a_1(x - x_0) \tag{7}$$

$$y_2 = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \tag{8}$$

Aitken Interpolation Method:

$$P_{o,k}(x) = \frac{1}{x_k - x_o}\begin{vmatrix} y_o & x_o - x \\ y_k & x_k - x \end{vmatrix} \tag{9}$$

$$P_{o,1,2}(x) = \frac{1}{x_2 - x_1}\begin{vmatrix} P_{o,1}(x) & x_1 - x \\ P_{o,2}(x) & x_2 - x \end{vmatrix} \tag{10}$$

$$y_n = P_{o,1,2,\ldots,n}(x) = \frac{1}{x_n - x_{n-1}}\begin{vmatrix} P_{o,1,\ldots,(n-1)}(x) & x_{n-1} - x \\ P_{o,1,\ldots,(n-2),n}(x) & x_n - x \end{vmatrix} \tag{11}$$

Examples: We will check the effectiveness of the present technique (3). First numerical comparison for the following test examples taken in [1].

Example 1: Solve $\frac{dy}{dx} = 1 + y^2(x), \quad y(0) = 0,$

By taking the step h = 0.01, which has the exact solution as: $y = \tan(x)$.
First by using Newton's interpolation, we have

$$a_0 = 0 = y_0, \quad a_1 = \frac{f(x_1 - x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,0} = 1$$

$$y_1 = 0 + 1(0.01 - 0) = 0.01, \quad a_2 = \frac{\frac{f(x_2 - x_1)}{(x_2 - x_1)} - \frac{f(x_1 - x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} = \frac{[\frac{dy}{dx}]_{0.01,0.01} - [\frac{dy}{dx}]_{0,0}}{0.02 - 0} = 0.005$$

$$y_2 = 0 + 1(0.02 - 0) + 0.005(0.02 - 0)(0.02 - 0.01) = 0.020001$$

Now, forming linear and quadratic using Aitken Method

$P_{0,1}(x) = x$

$P_{0,2}(x) = 1.00005xx$

$P_{0,1,2}(x) = 0.005x^2 + 0.99995x$

Hence, we can take the approximation solution of linear and quadratic using Aitken Method, if we take quadratic using Aitken Method, we find the same solution givenby Vinod M, Dimple R [4], Table 1.

Example 2: Solve $\frac{dy}{dx} = 1 + y^2(x), \quad y(0) = 0$

By taking the step h=0.01, which has the exact solution as: $y = \frac{c^{2x} - 2}{c^{2x} + 2}$

First by using Newton's interpolation, we have

Table 1: Solution of $\frac{dy}{dx} = 1 + y^2(x), \quad y(0) = 0$

x	Combined Newton's Interpolation and Aitken	Vinod M and Dimple R method	s
0	0	0	0
0.01	0.010000000	0.010000083	0.010000333
0.02	0.020001000	0.020000667	0.020002667
0.03	0.030003000	0.03000225	0.030009003
0.04	0.040006000	0.040005334	0.040021347
0.05	0.050010000	0.050010419	0.050041708
0.06	0.060015000	0.060018006	0.060072104
0.07	0.070021000	0.070028597	0.070114558
0.08	0.080028000	0.080042694	0.080171105
0.09	0.090036000	0.090060799	0.090243790
0.1	0.100045000	0.100083417	0.100334672

Table 2: Solution of $\frac{dy}{dx} = 1 + y^2(x), \quad y(0) = 0$

x	Combined Newton's Interpolation and Aitken	Vinod M and Dimple R method	s
0	0	0	0.0000000E+00
0.01	0.010000000	0.009999917	2.4998750E-07
0.02	0.020001000	0.019999333	1.9996001E-06
0.03	0.030003000	0.02999775	6.7469637E-06
0.04	0.040006000	0.039994668	1.5987209E-05
0.05	0.050010000	0.049989586	3.1210979E-05
0.06	0.060015000	0.059982006	5.3902948E-05
0.07	0.070021000	0.069971431	8.5540349E-05
0.08	0.080028000	0.079957361	1.2759151E-04
0.09	0.090036000	0.089939299	1.8151442E-04
0.1	0.100045000	0.09991675	2.4875529E-04

Now, forming linear and quadratic using Aitken Method

$P_{0,1}(x) = x$

$P_{0,2}(x) = 1.00005xx$

$P_{0,1,2}(x) = 0.005x^2 + 0.99995x$

Hence, we can take the approximation solution of linear and quadratic using Aitken Method, if we take quadratic using Aitken Method, we find the same solution given by Vinod M, Dimple R [4], Table 2.

Example 3: Solve $\frac{dy}{dx} = 1 - y^2(x) + 2y(x), \quad y(0) = 0$

By taking the step h = 0.01

First by using Newton's interpolation, we have

$$a_0 = 0.5 = y_0, \quad a_1 = \frac{f(x_1 - x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,0.5} = 0.5$$

$$y_1 = 0.5 + 0.5(0.01 - 0) = 0.505, \quad a_2 = \frac{\frac{f(x_2 - x_1)}{(x_2 - x_1)} - \frac{f(x_1 - x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} = \frac{[\frac{dy}{dx}]_{0.01,0.505} - [\frac{dy}{dx}]_{0,0.5}}{0.02 - 0} = -0.25$$

$$y_2 = 0.5 + 0.5(0.02 - 0) - 0.25(0.02 - 0)(0.02 - 0.01) = 0.50995$$

Table 3: Solution of $\frac{dy}{dx} = 1 - y^2(x) + 2y(x), \quad y(0) = 0$

x	Combined Newton's Interpolation and Aitken	Vinod M and Dimple R method	Taylor Series method	ADM	HPM
0	0	0	0	0	0
0.01	0.010000000	0.010100585	0.010100330	0.010100330	0.010100330
0.02	0.020199000	0.020404690	0.020402612	0.020402612	0.020402612
0.03	0.030597000	0.030915863	0.030908719	0.030908719	0.030908719
0.04	0.041194000	0.041637666	0.041620432	0.041620431	0.041620432
0.05	0.051990000	0.052573672	0.052539435	0.052539435	0.052539435
0.06	0.062985000	0.063727448	0.063667310	0.063667309	0.063667310
0.07	0.074179000	0.075102547	0.075005529	0.075005526	0.075005529
0.08	0.085572000	0.086702496	0.086555447	0.086555443	0.086555447
0.09	0.097164000	0.098530785	0.098318300	0.098318293	0.098318300
0.1	0.108955000	0.110590857	0.110295195	0.110295185	0.110295195

Now, forming linear and quadratic using Aitken Method

$P_{0,1}(x) = x$

$P_{0,2}(x) = 1.00995xx$

$P_{0,1,2}(x) = 0.995x^2 + 0.99005x$

Hence, we can take the approximation solution of linear and quadratic using Aitken Method, if we take quadratic using Aitken Method, we find the same solution given by Vinod M, Dimple R [4], Table 3.

Example 4: Solve $\frac{dy}{dx} = 1 - y^2(x) + 2y(x), \quad y(0) = 0, \quad x\ [0, 20]$

The exact solution is [35]

$$y(x) = 1 + \sqrt{2}\tanh\ (\sqrt{2x} + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right))$$

By taking the step h = 2 and by using Newton's interpolation, we have

$$a_0 = 0 = y_0, \quad a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,0} = 1$$

$$y_1 = 0 + 1.(2-0) = 2, \quad a_2 = \frac{[\frac{dy}{dx}]_{2,2} - [\frac{dy}{dx}]_{0,0}}{2-0} = 0, \quad y_2 = 0 + 1(2-0) = 2$$

Now, forming linear and quadratic using Aitken Method

$P_{0,1}(x) = x$

$P_{0,2}(x) = 0.5x$

$P_{0,1,2}(x) = -0.25x^2 + 1.5x$

Table 4: Solution of $\frac{dy}{dx} = 1 - y^2(x) + 2y(x), \quad y(0) = 0, x \; [0, 20]$

x	Exact solution	2-Iterate MVIM	3-Iterate VIM	Combined Newton's Interpolation and Aitken *for h=2*
2	2.3577716530	2.3592420980	-286352.73097900	2.0000000000
4	2.4140123820	2.4140330560	-9.0657980428E19	2.0000000000
6	2.4142128590	2.4142130890	-7.3332282199E33	-3.000000000
8	2.4142135600	2.4142136400	-5.7930892793E47	-4.000000000
10	2.4142135620	2.4142136400	-4.5744317440E61	-10.00000000
12	2.4142135620	2.4142136400	-3.6121072512E75	-18.00000000
14	2.4142135620	2.4142136560	-2.8522268182E89	-28.00000000
16	2.4142135620	2.4142136940	-2.2522027269E103	-40.00000000
18	2.4142135620	2.4142136830	-1.7784059425E117	-54.00000000

Hence, we can take the approximation solution of quadratic using Aitken Method, we find the solution given by Table 4. It is very far from the exact solution and also far from the solution given by the (VIM) method [35].

To solve this problem, for some points only, Belal B. [35] took the step h = 0.2 in VIM method. by the Hybrid method taking h small, h = 0.2 we can see the solution.

First by using Newton's interpolation, we have

$$a_0 = 0 = y_0, \quad a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,0} = 1$$

$$y_1 = 0 + 1(0.2 - 0) = 0.2, \quad a_2 = \frac{[\frac{dy}{dx}]_{0.2,0.2} - [\frac{dy}{dx}]_{0,0}}{0.4 - 0} = 0.9$$

$$y_2 = 0 + 1(0.4 - 0) + 0.9(0.4 - 0)(0.4 - 0.2) = 0.472$$

Now, forming linear and quadratic using Hybrid Method

$P_{0,1}(x) = x$

$P_{0,2}(x) = 1.18x$

$P_{0,1,2}(x) = 0.9x^2 + 0.82x$

Hence, we can take the approximation solution of quadratic using Hybrid Method, we find the solution given by Table 2. It is also far from the exact solution and also far from the solution given by the (VIM) method [35]. Finally by taking the step h=0.01, we have the following linear and quadratic using Hybrid Method

$$a_0 = 0.5 = y_0, \quad a_1 = \frac{f(x_1 - x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,0.5} = 0.5$$

$$y_1 = 0.5 + 0.5(0.01 - 0) = 0.505, \quad a_2 = a_2 = \frac{\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} = \frac{[\frac{dy}{dx}]_{0.01,0.505} - [\frac{dy}{dx}]_{0,0.5}}{0.02 - 0} = -0.25$$

$$y_2 = 0.5 + 0.5(0.02 - 0) - 0.25(0.02 - 0)(0.02 - 0.01) = 0.50995$$

Table 5: Solution of $\frac{dy}{dx} = 1 - y^2(x) + 2y(x), \quad y(0) = 0, x \ [0, 2]$

x	Exact solution	2-Iterate MVIM	3-Iterate VIM	Combined Newton's Interpolation and Aitken *for h=0.2*	Combined Newton's Interpolation and Aitken *for h=0.01*
0.2	0.2419767992	0.2396149017	0.2419778327	0.200000000	0.237810000
0.4	0.5678121656	0.5626231618	0.5678455132	0.472000000	0.555220000
0.6	0.9535662155	0.9468409011	0.9536660329	0.816000000	0.952230000
0.8	1.3463636550	1.3405640980	1.3463791062	1.232000000	1.428840000
1.0	1.6894983900	1.6863821450	1.6860271032	1.720000000	1.985050000
1.2	1.9513601180	1.9509491870	1.9150510260	2.280000000	2.620860000
1.4	2.1313266100	2.1325827440	2.1791315021	2.912000000	3.336270000
1.6	2.2462859590	2.2481414290	-50.98229780	3.616000000	4.131280000
1.8	2.3163247370	2.3181237490	-5338.782860	4.392000000	5.005890000
2.0	2.3577716530	2.3592420980	-286352.7325	5.240000000	7.940200000

Now, forming linear and quadratic using Aitken Method

$P_{0,1}(x) = x$

$P_{0,2}(x) = 1.00995x$

$P_{0,1,2}(x) = 0.995x^2 + 0.99005x$

Now let's use this approximation to get the solution for x=0.2, 0,4, 0.6,….

We find the results in Table 5. Which shows us a significant improvement in the solution, but not in all points, only in the first three points and therefore it can be considered useful to get the solution in these three points only.

CONCLUSIONS

In this work, we have been solve the Riccati nonlinear first order differential equation by the combined of Newton's interpolation and Aitken's method, we compare the result for some examples with Vinod M and Dimple R method, we find the same results given by Vinod M and Dimple R. After, we have compared the result for one example with exact solution, variational iteration method (VIM) and multistage variational method (MVIM), we found that using this combined method is not generally effective, not stable and this method can be improved by minimizing the step h for the solution interval and obtaining the approximation relationship, then using it in a limited number of first points of the solution interval.

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