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Helices in a Lorentzian 6-Space

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Abstract: The aim of this paper is to determine the Frenet-Serret invariants of non-null curves in a Lorentzian 6-space. First, we introduce a vector which is derived from five vectors, by this way; we present a method to calculate Frenet-Serret invariants of the non-null helices. Additionally, an algebraic example of our method is presented.

Key words: Classical Differential Geometry · Lorentzian space · Frenet-Serret Invariants · Non-null Curves · Helix

INTRODUCTION

In the local differential geometry, the Frenet-Serret frame has basic importance to investigate characterizations of the curves. For instance, curvature functions give us the behavior of the curves. Because in the local differential geometry, we think of curves as a geometric set of points or locus, therefore, investigating Frenet-Serret frame of the curve is a classical aim to determine its behavior. There is an extensive literature on the subject, for instance, in [1] the author presented a method to calculate Frenet-Serret apparatus of the regular curves in the Euclidean 4-space. Thereafter in [2], the author adapted this method to spacelike curves of Minkowski 4-space according to signature (+,+,+,–) and investigated spherical images of such curves. However, in the existing literature the works were commonly based on the spacelike and timelike curves of Minkowski space-time with signature (–,+,+,+). Since, in [3] and [4], respectively, the mentioned method (originally expressed as in [1]) was adapted and developed for spacelike and timelike curves of Minkowski space-time. Recently, in [6], the same idea is studied for non-null curves of Lorentzian 5-space by the spirit of the paper [6]. A. Einstein's theory opened a door to using new geometries and thus the researchers discovered a bridge between modern differential geometry and mathematical physics. Generally they used the concepts of a mapping and a curve. For instance, a "particle" in special relativity means a curve with a timelike unitary tangent vector, for details [7]; and in another work, it has been also observed that third curvature is important [8]. Thereafter null cases were also studied to make a understanding tool of general relativity as a dynamical theory and the Frenet-Serret formalism and by this way black holes were investigated in five and six dimensional space by considering a timelike curve [9, 10]. However, this paper did not include an explicit calculation of Frenet-Serret frame with a vector derived from a generalization of cross product in Lorentzian space. A curve of constant slope or general helix is defined by the property that the tangent lines make a constant angle with a fixed direction. A necessary and sufficient condition that a curve to be general helix in Minkowski 3-space is that ratio of curvature to torsion be constant [11]. Indeed, a helix is a special case of the general helix. If both curvature and torsion are non-zero constants, it is called a helix or only a W-curve [12, 13]. Helices arise in nanosprings, carbon nanotubes, α – helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA [14]. This fact was first published by Watson and Crick in 1952 [15]. They constructed a molecular model of DNA in which there were two

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complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiesterase bonds [14, 16, 17]. In this work, we extend the notion of calculate Frenet-Serret apparatus of helices (W-curves, i.e.) considering the paper [18] which contains regular observations of helices in six dimensional Euclidean space. Additionally, we express an example of our main results. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

Preliminaries: To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbf{L}^6$ are briefly presented, a complete treatment can be found in [19]. Lorentzian space $\mathbf{L}^6$ is a pseudo-Euclidean space $\mathbf{E}^6$ provided with the standard flat metric given by:

$$g = -dx_1^2 + \sum_{i=2}^{6} dx_i^2$$

where $(x_1, x_2, ..., x_6)$ is a rectangular coordinate system in $\mathbf{L}^6$. Since g is an indefinite metric, recall that a vector $\vec{v} \in \mathbf{L}^6$ can have one of the three causal characters; it can be spacelike if $g(\vec{v},\vec{v}) > 0$ or $\vec{v} = 0$, timelike if $g(\vec{v},\vec{v}) < 0$ and null (lightlike) if $g(\vec{v},\vec{v}) = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\gamma} = \vec{\gamma}(s)$ in $\mathbf{L}^6$ can be locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{\gamma}'(s)$ are respectively spacelike, timelike or null. Also, we recall that the norm of a vector $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{|g(\vec{v},\vec{v})|}$. Therefore, $\vec{v}$ is a unit vector if $g(\vec{v},\vec{v}) = \pm 1$. Next, vectors $\vec{v}, \vec{w}$ in $\mathbf{L}^6$ are said to be orthogonal if $g(\vec{v},\vec{w}) = 0$. The velocity of the curve $\vec{\gamma}$ is given by $\|\vec{\gamma}'\|$. Thus, a spacelike or a timelike curve $\vec{\gamma}$ is said to be parametrized by arc length function s, if $g(\vec{\gamma}',\vec{\gamma}') = \pm 1$.

Denote by $\left\{\vec{V}_1(s), \vec{V}_2(s), \vec{V}_3(s), \vec{V}_4(s), \vec{V}_5(s), \vec{V}_6(s)\right\}$ the moving Frenet-Serret frame along the curve $\vec{\gamma}(s)$ in the space $\mathbf{L}^6$. Then, for a non-null unit speed curve of $\mathbf{L}^6$, the following Frenet-Serret equations are given in [20, 21]:

$$\begin{bmatrix} \vec{V}_1' \\ \vec{V}_2' \\ \vec{V}_3' \\ \vec{V}_4' \\ \vec{V}_5' \\ \vec{V}_6' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & 0 & 0 \\ -\varepsilon_0\varepsilon_1\kappa_1 & 0 & \kappa_2 & 0 & 0 & 0 \\ 0 & -\varepsilon_1\varepsilon_2\kappa_2 & 0 & \kappa_3 & 0 & 0 \\ 0 & 0 & -\varepsilon_2\varepsilon_3\kappa_3 & 0 & \kappa_4 & 0 \\ 0 & 0 & 0 & -\varepsilon_3\varepsilon_4\kappa_4 & 0 & \kappa_5 \\ 0 & 0 & 0 & 0 & -\varepsilon_4\varepsilon_5\kappa_5 & 0 \end{bmatrix} \begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \\ \vec{V}_4 \\ \vec{V}_5 \\ \vec{V}_6 \end{bmatrix}$$

where $g(\vec{V}_j,\vec{V}_j) = \varepsilon_{j-1} = \mp 1$ for $1 \leq j \leq 6$, according to character of frame vector. Here, κ_i are the curvature functions, as well; and we shall call the set whose elements are curvature functions and Frenet-Serret frame vector fields as *Frenet-Serret invariants* of the curves. Here, recall that, a curve is called a *helix* if it has constant Frenet-Serret curvatures.

Main Results: In this section, first, we define a vector as follows:
Let $\vec{a} = (a_1, a_2, ..., a_6)$, $\vec{b} = (b_1, b_2, ..., b_6)$, $\vec{c} = (c_1, c_2, ..., c_6)$, $\vec{d} = (d_1, d_2, ..., d_6)$ and $\vec{f} = (f_1, f_2, ..., f_6)$ be non-null vectors in $\mathbf{L}^6$. We define a vector in $\mathbf{L}^6$ with the determinant:

$$\vec{\Psi} = \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} = -\begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 & \vec{e}_5 & \vec{e}_6 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{vmatrix},$$

where $\vec{e}_i$ for $1 \le i \le 6$ are coordinate direction (basis) vectors of $\mathbf{L}^6$ which satisfy:

$$\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 = \vec{e}_6,\ \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_6 = \vec{e}_1,$$
$$\vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_6 \wedge \vec{e}_1 = \vec{e}_2,\ \vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_6 \wedge \vec{e}_1 \wedge \vec{e}_2 = -\vec{e}_3,\ \vec{e}_3 \wedge \vec{e}_2 \wedge \vec{e}_1 \wedge \vec{e}_5 \wedge \vec{e}_6 = \vec{e}_4.$$

Via this product it is safe to report that:

$$\begin{aligned}\left\langle \vec{a}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \right\rangle &= \left\langle \vec{b}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \right\rangle \\ &= \left\langle \vec{c}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \right\rangle \\ &= \left\langle \vec{d}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \right\rangle \\ &= \left\langle \vec{f}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \wedge \vec{f} \right\rangle = 0.\end{aligned}$$

Let $\vec{\gamma} = \vec{\gamma}(s)$ be a non-null unit speed helix in $\mathbf{L}^6$. One can write the following differentiations with respect to s:

$$\begin{cases}\vec{\gamma}' = \vec{V}_1, \\ \vec{\gamma}'' = \kappa_1 \vec{V}_2, \\ \vec{\gamma}''' = -\varepsilon_0 \varepsilon_1 \kappa_1^2 \vec{V}_1 + \kappa_1' \vec{V}_2 + \kappa_1 \kappa_2 \vec{V}_3, \\ \vec{\gamma}^{(IV)} = (-\varepsilon_0 \varepsilon_1 \kappa_1^3 - \varepsilon_1 \varepsilon_2 \kappa_1 \kappa_2^2) \vec{V}_2 + (\kappa_1 \kappa_2 \kappa_3) \vec{V}_4 \\ \vec{\gamma}^{(V)} = \varepsilon_0 \kappa_1^2 (\varepsilon_0 \kappa_1^2 + \varepsilon_2 \kappa_2^2) \vec{V}_1 - \kappa_1 \kappa_2 (\varepsilon_0 \varepsilon_1 \kappa_1^2 + \varepsilon_0 \varepsilon_2 \kappa_2^2 + \varepsilon_1 \varepsilon_2 \kappa_3^2) \vec{V}_3 + (\kappa_1 \kappa_2 \kappa_3 \kappa_4) \vec{V}_5 \\ \vec{\gamma}^{(VI)} = (...)\vec{V}_1 + (...)\vec{V}_2 + (...)\vec{V}_3 + (...)\vec{V}_4 + (...)\vec{V}_5 + (\kappa_1 \kappa_2 \kappa_3 \kappa_4 \kappa_5) \vec{V}_6\end{cases} \tag{1}$$

From first equation of (1), we know first vector field of Frenet-Serret frame. Thereafter, by means of $(1)_2$, we express:

$$\left\| \vec{\gamma}'' \right\| = \kappa_1 \tag{2}$$

and

$$\vec{V}_2 = \frac{\vec{\gamma}''}{\kappa_1}. \tag{3}$$

Then, taking the norm of both sides of (1), we write the second curvature as:

$$\kappa_2 = \sqrt{\left| \frac{1}{\varepsilon_2} \left(\frac{\left\| \vec{\gamma}''' \right\|}{\left\| \vec{\gamma}'' \right\|} \right)^2 - \varepsilon_0 \left\| \vec{\gamma}'' \right\|^2 \right|}. \tag{4}$$

Using (4) and considering (1), we have the third vector field:

$$\vec{V}_3 = \frac{\vec{\gamma}''' + \varepsilon_0 \varepsilon_1 \left\| \vec{\gamma}'' \right\|^2 \vec{\gamma}'}{\sqrt{\left| \frac{1}{\varepsilon_2} \left(\left\| \vec{\gamma}''' \right\|^2 - \varepsilon_0 \left\| \vec{\gamma}'' \right\|^4 \right) \right|}}. \tag{5}$$

To calculate the third curvature and the fourth vector field, let us form:

$$\left\| \vec{\gamma}'' \right\|^2 \vec{\gamma}^{(IV)} + \varepsilon_1 \left\| \vec{\gamma}''' \right\|^2 \vec{\gamma}' = \kappa_1^3 \kappa_2 \kappa_3 \vec{V}_4. \tag{6}$$

Equation (6) yields, respectively,

$$\vec{V}_4 = \frac{\left\|\vec{\gamma}''\right\|^2 \vec{\gamma}^{(IV)} + \varepsilon_1 \left\|\vec{\gamma}'''\right\|^2 \vec{\gamma}'}{\left\| \left\|\vec{\gamma}''\right\|^2 \vec{\gamma}^{(IV)} + \varepsilon_1 \left\|\vec{\gamma}'''\right\|^2 \vec{\gamma}' \right\|} \tag{7}$$

and

$$k_3 = \frac{\left\| \left\|\vec{\gamma}''\right\|^2 \vec{\gamma}^{(IV)} + \varepsilon_1 \left\|\vec{\gamma}'''\right\|^2 \vec{\gamma}' \right\|}{\sqrt{\left| \frac{1}{\varepsilon_2} (\left\|\vec{\gamma}'''\right\|^2 - \varepsilon_0 \left\|\vec{\gamma}''\right\|^4) \right|}}. \tag{8}$$

Now, let us calculate the vector of $\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \wedge \vec{\gamma}^{(V)}$ according to frame $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5, \vec{V}_6\right\}$. This expression follows that:

$$\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \wedge \vec{\gamma}^{(V)} = \varepsilon_5 k_1^3 k_2^3 k_3^2 k_4 \vec{V}_6. \tag{9}$$

Using (9), we easily have $\vec{V}_6$ and the fourth curvature of the curve $\vec{\gamma} = \vec{\gamma}(s)$, respectively,

$$\vec{V}_6 = \eta \frac{\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \wedge \vec{\gamma}^{(V)}}{\left\|\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \wedge \vec{\gamma}^{(V)}\right\|}, \tag{10}$$

and

$$k_4 = \frac{\left\|\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \wedge \vec{\gamma}^{(V)}\right\|}{\left\| \left\|\vec{\gamma}''\right\|^2 \vec{\gamma}^{(IV)} + \varepsilon_1 \left\|\vec{\gamma}'''\right\|^2 \vec{\gamma}' \right\|^2 \sqrt{\left| \frac{1}{\varepsilon_2} (\left\|\vec{\gamma}'''\right\|^2 - \varepsilon_0 \left\|\vec{\gamma}''\right\|^4) \right|}}. \tag{11}$$

η in the expression (11) is taken ± 1 to make $+1$ determinant of $\left[\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5, \vec{V}_6\right]$ matrix. By this way, Frenet-Serret frame is positively oriented. Using the inner product of (1) and (10), we obtain $g(\vec{\gamma}^{(VI)}, \vec{V}_6) = \varepsilon_5 k_1 k_2 k_3 k_4 k_5.$ Since the fifth curvature of $\vec{\gamma}(s)$ as:

$$k_5 = \frac{g(\vec{\gamma}^{(VI)}, \vec{V}_6) \left\| \left\|\vec{\gamma}''\right\|^2 \vec{\gamma}^{(IV)} + \varepsilon_1 \left\|\vec{\gamma}'''\right\|^2 \vec{\gamma}' \right\| \sqrt{\left| \frac{1}{\varepsilon_2} (\left\|\vec{\gamma}'''\right\|^2 - \varepsilon_0 \left\|\vec{\gamma}''\right\|^4) \right|}}{\varepsilon_5 \left\|\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \wedge \vec{\gamma}^{(V)}\right\|}.$$

Finally, the vector $\vec{V}_5 = \eta . \vec{V}_3 \wedge \vec{V}_2 \wedge \vec{V}_1 \wedge \vec{V}_5 \wedge \vec{V}_6$ gives us the fifth vector field of the Frenet-Serret frame. Thus, we have calculated Frenet-Serret invariants of the curve $\vec{\gamma} = \vec{\gamma}(s)$. Moreover, suffice it to say that $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5, \vec{V}_6\right\}$ is an orthonormal frame of $\mathbf{L}^6$.

An Algebraic Example of the Presented Method: In this section we present an example of our main results. Let us consider the following timelike curve:

$$\beta = \beta(s) = \left(2\sinh s, \quad 2\cosh s, \quad 2\sin\frac{s}{2}, \quad -2\cos\frac{s}{2}, \quad 2\sin\frac{s}{\sqrt{2}}, \quad -2\cos\frac{s}{\sqrt{2}} \right). \tag{11}$$

Differentiating (11) with respect to *s*, we immediately arrive at:

$$V_1 = \left(2\cosh s,\quad 2\sinh s,\quad \cos\frac{s}{2},\quad -\sin\frac{s}{2},\quad \sqrt{2}\cos\frac{s}{\sqrt{2}},\quad \sqrt{2}\sin\frac{s}{\sqrt{2}}\right).$$

By virtue of (2), we have $\kappa_1 = \frac{\sqrt{21}}{2}$. One more differentiating of (11) and using (3), we find the second vector field of Frenet-Serret frame:

$$V_2 = \left(\frac{4\sinh s}{\sqrt{21}},\quad \frac{4\cosh s}{\sqrt{21}},\quad -\frac{\sin\frac{s}{2}}{\sqrt{21}},\quad \frac{\cos s}{\sqrt{21}},\quad -\frac{2\sin\frac{s}{\sqrt{2}}}{\sqrt{21}},\quad \frac{2\cos\frac{s}{\sqrt{2}}}{\sqrt{21}}\right).$$

By the equation (4), we obtain the second curvature $\kappa_2 = \sqrt{\frac{193}{42}}$. We find the third vector field by the equation (5):

$$V_3 = \frac{1}{\sqrt{193}}\left(-17\sqrt{2}\cosh s,\quad -17\sqrt{2}\sinh s, -11\sqrt{2}\cos\frac{s}{2}, -11\sqrt{2}\sin\frac{s}{2}, -23\cos\frac{s}{\sqrt{2}}, -23\sin\frac{s}{\sqrt{2}}\right).$$

By the equations (7) and (8), we express, respectively $\kappa_3 = \sqrt{\frac{677}{8106}}$ and:

$$V_4 = \frac{1}{\sqrt{4217}}\left(58\sinh s,\ 58\cosh s,\ 38\sin\frac{s}{2}, -38\cos\frac{s}{2},\ 97\sin\frac{s}{\sqrt{2}},\ 97\cos\frac{s}{\sqrt{2}}\right).$$

Forming the vector $\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \wedge \vec{\gamma}^{(V)}$, we have the sixth vector field:

$$V_6 = \frac{1}{\sqrt{677}}\left(\sinh s,\ \cosh s,\ 24\sin\frac{s}{2}, -24\cos\frac{s}{2},\ -10\sin\frac{s}{\sqrt{2}},\ 10\cos\frac{s}{\sqrt{2}}\right).$$

By the equation (11), we have that $\kappa_4 = 15\sqrt{\frac{21}{130661}}$. Finally, the vector: $\vec{V}_3 \wedge \vec{V}_2 \wedge \vec{V}_1 \wedge \vec{V}_5 \wedge \vec{V}_6$ gives us the fifth vector field and the fifth curvature:

$$V_5 = \frac{1}{\sqrt{193}}\left(-\cosh s,\ -\sinh s, -12\cos\frac{s}{2}, -12\sin\frac{s}{2},\ 5\sqrt{2}\cos\frac{s}{\sqrt{2}},\ 5\sqrt{2}\sin\frac{s}{\sqrt{2}}\right)$$

and $\kappa_5 = \sqrt{\frac{193}{677}}$. It is safe to report that the curve is a timelike helix and $\{\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5, \vec{V}_6\}$ is an orthonormal frame of $\mathbf{L}^6$.

CONCLUSIONS

In Minkowski space, the timelike helices are important because they represent the trajectories of classical charged particles under an arbitrary constant electromagnetic field, if their motion is given by the Lorentz equation. Then it is natural to generalize the study of helices to spaces of higher dimensions, for example, in Lorentzian 6-spaces, showing an efficient method to determine the Frenet-Serret invariants of non-null helices in such 6-spaces, thus in this paper it was exhibited an approach to construct these invariants.

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