Middle-East Journal of Scientific Research 28 (2): 142-149, 2020
ISSN 1990-9233
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DOI: 10.5829/idosi.mejsr.2020.142.149

A New Hybrid Sumudu Transform With Homotopy Perturbation Method For Solving Boundary Value Problems

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Abstract: In this paper, a combination between a sumudu transform (ST) and the homotopy perturbation method (HPM) is presented. This combination allows to obtain approximate and exact solutions for linear and nonlinear boundary value problems. The results obtained by the proposed hybrid method (ST-HPM) reveal that is very effective and convenient.

Key words: Sumudu transform · Homotopy perturbation method · Boundary value problems · Nonlinear ordinary differential equation

INTRODUCTION

The homotopy perturbation method proposed first by J. He (2000) [1-5] for solving linear and nonlinear boundary value problems. This method was applied to solve nonlinear partial differential equation [6], nonlinear singular Lane–Emden equations [7], nonlinear fractional Kolmogorov–Petrovskii–Piskunov equations [8] and many other subjects. Sh. Javeed et al. successfully applied the HPM for solving fractional order differential equations [9].

E.E. Eladdad and E.A. Tarif proposed a hybridization of a new integral transform with homotopy perturbation method to obtain analytic and numerical solutions for solving systems of partial differential equations [10].

On the other hand, the sumudu transform was proposed by Watugala (1993) [11, 12]. The ST operator was denoted by S[.] which it has been defined by the following integral equation.

$$S[f(t)] = G(v) = \frac{1}{v}\int_0^\infty e^{-\frac{t}{v}} f(t)\,dt, \qquad t \geq 0$$

where

$$S\left[f^{(n)}(t)\right] = \frac{G(v)}{v^n} - \sum_{k=0}^{n-1} v^{-n+k} f^{(k)}(0), \qquad n \geq 1$$

The Objective of this paper is to construct the homotopy for nonlinear terms and hybridization the basic HPM with ST.

A Hybrid Method (ST-HPM): Consider the following nonlinear ordinary differential equation.

$$u^{(n)}(x) = f\left(x, e^{g(u(x))}\right) \quad 0 \leq x \leq 1 \tag{1}$$

With the boundary conditions
$u^{(p)}(0) = \alpha_p,\ u^{(q)}(1) = \beta_q;\quad \alpha_p,\quad \beta_q = const$

The homotopy of Eq. (1) can be written as follows.

$$u^{(n)}(x) = f\left(x, e^{g(p.u(x))}\right) \tag{2}$$

where $p \in [0,1]$ is an embedding parameter

Taking the ST on Eq. (2), yields

$$S\left[u^{(n)}(x)\right] = S\left[f\left(x, e^{g(p.u(x))}\right)\right] \tag{3}$$

Then,

$$\frac{S[u]}{v^n} - \sum_{k=0}^{n-1} v^{-n+k} u^{(k)}(0) = S\left[f\left(x, e^{g(p.u(x))}\right)\right] \tag{4}$$

Then,

$$S[u] = v^n \sum_{k=0}^{n-1} v^{-n+k} u^{(k)}(0) + v^n S\left[f\left(x, e^{g(p.u(x))}\right)\right] \tag{5}$$

According to the HPM [2] the solution of Eq. (5) can be written as a power series in.

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$$u=\sum_{i=0}^{\infty}p^i u_i \tag{6}$$

Substituting Eq. (6) into Eq. (5), yields,

$$S\left[\sum_{i=0}^{\infty}p^i u_i\right]=v^n\sum_{k=0}^{n-1}v^{-n+k}u^{(k)}(0)+v^n S\left[f\left(x,e^{g\left(\sum_{i=0}^{\infty}p^{i+1}u_i\right)}\right)\right] \tag{7}$$

Comparing coefficients of terms with identical powers of p in Eq. (7) and taking the inverse ST and setting $p = 1$, we have the approximate solution of Eq. (1) as follows.

$$u(x)=\sum_{i=0}^{\infty}u_i(x) \tag{8}$$

The series in Eq. (8) is convergent in most cases and the convergence rate of the series depends on the nonlinear operator [1].

Illustrative Examples: In this section, we apply our method (ST-HPM) for solving some examples.

Example 1: Consider the following nonlinear ordinary differential equation.

$$u''-xe^u=0;\quad u=u(x)\quad 0\le x\le 1 \tag{9}$$

Subject to the boundary conditions.

$u(0) = 0, \qquad u(1) = 0$

The homotopy of Eq. (9) can be written as follows.

$$u''-xe^{pu}=0 \tag{10}$$

Developing e^{pu} by Maclaurin series, then Eq. (10) can be written as follows;

$$u''=x\left(1+pu+\frac{p^2u^2}{2}+\frac{p^3u^3}{6}+...\right) \tag{11}$$

Taking the ST on Eq. (11), yields,

$$S[u'']=S\left[x\left(1+pu+\frac{p^2u^2}{2}+\frac{p^3u^3}{6}+...\right)\right] \tag{12}$$

Then,

$$\frac{S(u)}{v^2}-\frac{u(0)}{v^2}-\frac{u'(0)}{v}=S\left[x\left(1+pu+\frac{p^2u^2}{2}+\frac{p^3u^3}{6}+...\right)\right] \tag{13}$$

Or,

$$S(u)=\alpha v+v^2 S\left[x\left(1+pu+\frac{p^2u^2}{2}+\frac{p^3u^3}{6}+...\right)\right];\quad u'(0)=\alpha \tag{14}$$

Substituting Eq. (6) into Eq. (14), we get,

$$S\left(\sum_{i=0}^{\infty}p^i u_i\right)=\alpha v+v^2 S\left[x\left(1+\left(\sum_{i=0}^{\infty}p^{i+1}u_i\right)+\frac{1}{2}\left(\sum_{i=0}^{\infty}p^{i+1}u_i\right)^2+\frac{1}{6}\left(\sum_{i=0}^{\infty}p^{i+1}u_i\right)^3+...\right)\right] \tag{15}$$

Comparing coefficients of terms with identical powers of p in Eq. (15), leads to,

$$p^0: S(u_0) = \alpha v + v^2 S(x) \tag{16}$$

$$p^1: S(u_1) = v^2 S(x u_0) \tag{17}$$

$$p^2: S(u_2)=v^2 S\left(xu_1+\frac{x}{2}u_0^2\right) \tag{18}$$

$$p^3: S(u_3)=v^2 S\left(xu_2+xu_0u_1+\frac{x}{6}u_0^3\right) \tag{19}$$

$$p^4: S(u_4)=v^2 S\left(xu_3+xu_0u_2+\frac{x}{2}u_1^2+\frac{x}{2}u_0^2u_1\right) \tag{20}$$

Taking the inverse ST of Eqs. (16), (17), (18), (19) and (20), yields,

$$u_0=\alpha x+\frac{x^3}{6}$$

$$u_1=\frac{\alpha x^4}{12}+\frac{x^6}{180}$$

$$u_2 = \frac{\alpha^2 x^5}{40} + \frac{\alpha x^7}{168} + \frac{7x^9}{25920}$$

$$u_3 = \frac{\alpha^3 x^6}{180} + \frac{23\alpha^2 x^8}{6720} + \frac{11\alpha x^{10}}{25200} + \frac{17x^{12}}{1140480}$$

$$u_4 = \frac{13\alpha^3 x^9}{12960} + \frac{97\alpha^2 x^{11}}{316800} + \frac{3851\alpha x^{13}}{141523200} + \frac{2609x^{15}}{3592512000}$$

Then, the approximate solution of Eq. (9) with the boundary conditions is given by.

$$u = u_0 + u_1 + u_2 + ...$$

By the boundary condition $u(1) = 0$ and Maple program for $N = 3$, we have,

$$\alpha = -0.1589333052$$

For $N = 4$, we have,

$$\alpha = -0.1589422604$$

For $N = 5$, we have ,

$$\alpha = -0.1589423609$$

The numerical results were obtained in Table 1. In Table 1, we list the results obtained by the proposed method (ST-HPM) at $x = 0(0.1)1$. As can be seen from Table 1, the error decreased when the integer N is increased until $N = 5$.

The curve in Fig. 1 shows an approximate solution for example 1.

Table 1: Numerical results for Example 1

x	Error of ST-HPM ($N = 3$) $u_0 + u_1 + u_2$	Error of ST-HPM ($N = 4$) $u_0 + u_1 + u_2 + u_3$	Error of ST-HPM ($N = 5$) $u_0 + u_1 + u_2 + u_3 + u_4$
0	0	0	0
0.1	6.191 e - 8	2.403 e - 10	2.301 e - 10
0.2	7.724 e - 7	3.912 e - 9	2.450 e - 9
0.3	2.289 e - 6	3.924 e - 9	2.148 e - 8
0.4	1.313 e - 6	1.449 e - 7	1.339 e - 7
0.5	1.074 e - 5	6.581 e - 7	2.544 e - 7
0.6	4.469 e - 5	1.488 e - 6	2.946 e - 7
0.7	1.029 e - 4	1.286 e - 6	9.290 e - 8
0.8	1.606 e - 4	3.502 e - 6	3.785 e - 7
0.9	1.449 e - 4	1.740 e - 5	6.506 e - 7
1	7.438 e - 5	3.917 e - 5	6.042 e - 7

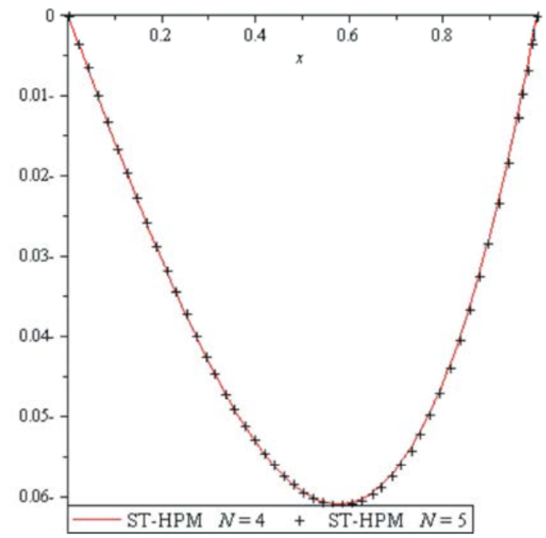

Fig. 1: The approximate solution by using (ST-HPM) for example 1

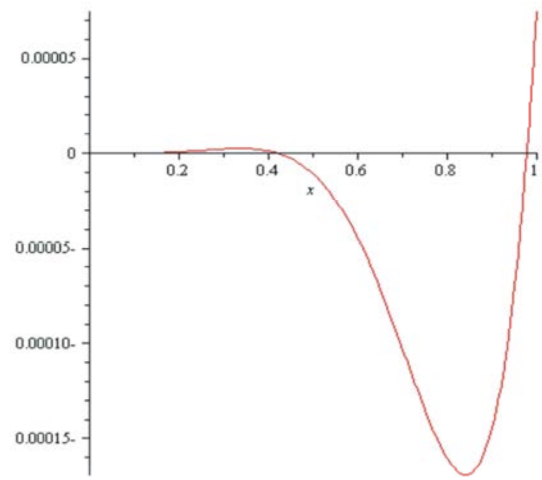

Fig. 2: It illustrates the error for $N = 3$

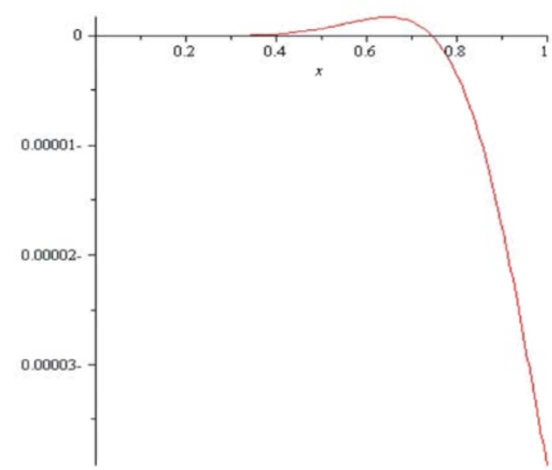

Fig. 3: It illustrates the error for $N = 4$

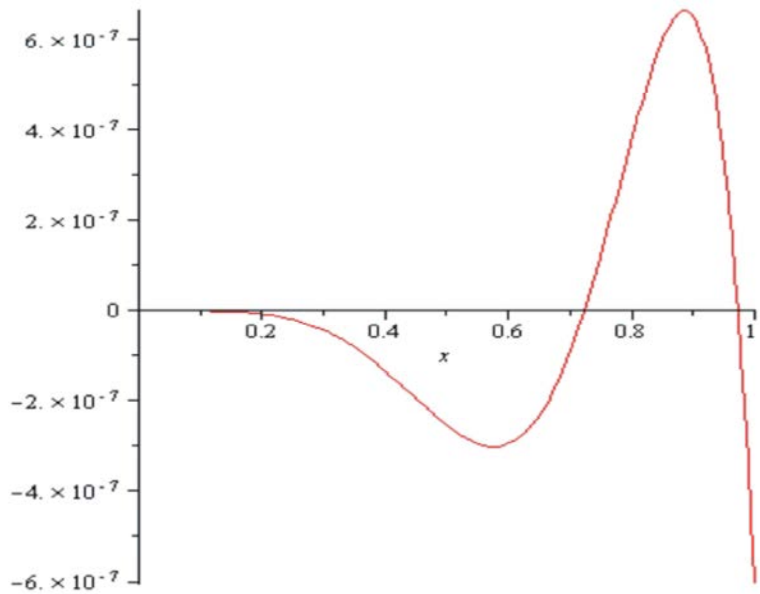

Fig. 4: It illustrates the error for $N = 5$

Example 2: Consider the following nonlinear ordinary differential equation.

$$u^{(4)} = e^{-u} - e^{-2u} = 0; \quad u = u(x) \quad 0 \le x \le 1 \tag{21}$$

Subject to the boundary conditions,

$$u(0) = 0, \quad u'(0) = 1, \quad u(1) = 0, \quad u'(1) = 1$$

The homotopy of Eq. (21) can be written as follows

$$u^{(4)} = e^{-pu} - e^{-2pu} = 0 \tag{22}$$

Developing e^{pu} by Maclaurin series, then Eq. (22) can be written as follows,

$$u^{(4)} = 2 - 3pu + \frac{5}{2}p^2u^2 - \frac{3}{2}p^3u^3 + ... \tag{23}$$

Taking the ST on Eq. (23), yields,

$$S\left[u^{(4)}\right] = S\left[2 - 3pu + \frac{5}{2}p^2u^2 - \frac{3}{2}p^3u^3 + ...\right] \tag{24}$$

Then,

$$\frac{s(u)}{v^4} - \frac{u(0)}{v^4} - \frac{u'(0)}{v^3} - \frac{u''(0)}{v^2} - \frac{u^{(3)}(0)}{v} = S\left[2 - 3pu + \frac{5}{2}p^2u^2 - \frac{3}{2}p^3u^3 + ...\right] \tag{25}$$

Or,

$$S(u) = v + \alpha v^2 + \beta v^3 + v^4 S\left[2 - 3pu + \frac{5}{2}p^2u^2 - \frac{3}{2}p^3u^3 + ...\right]; \quad \alpha = u''(0), \ \beta = u^{(3)}(0) \tag{26}$$

Substituting Eq. (6) into Eq. (26), we get,

$$S\left(\sum_{i=0}^{\infty} p^i u_i\right) = v + \alpha v^2 + \beta v^3 + v^4 S\left[2 - 3\left(\sum_{i=0}^{\infty} p^{i+1}u_i\right) + \frac{5}{2}\left(\sum_{i=0}^{\infty} p^{i+1}u_i\right)^2 - \frac{3}{2}\left(\sum_{i=0}^{\infty} p^{i+1}u_i\right)^3 + ...\right] \tag{27}$$

Comparing coefficients of terms with identical powers of p in Eq. (27), leads to,

$$p^0 : S(u_0) = v + \alpha v^2 + \beta v^3 + v^4 S(2) \tag{28}$$

$$p^1 : S(u_1) = v^4 S(-3u_0) \tag{29}$$

$$p^2 : S(u_2) = v^4 S\left(-3u_1 + \frac{5}{2}u_0^2\right) \tag{30}$$

$$p^3 : S(u_3) = v^4 S\left(-3u_2 + 5u_0u_1 - \frac{3}{2}u_0^3\right) \tag{31}$$

$$p^4 : S(u_4) = v^4 S\left(-3u_3 + 5u_0 u_2 + \frac{5}{2}u_1^2 - \frac{9}{2}u_0^2 u_1\right) \quad (32)$$

Taking the inverse ST of Eqs. (28), (29), (30), (31) and (32), yields,

$$u_0 = x + \frac{\alpha x^2}{2} + \frac{\beta x^3}{6} + \frac{x^4}{12}$$

$$u_1 = -\frac{x^5}{40} - \frac{\alpha x^6}{240} - \frac{\beta x^7}{1680} - \frac{x^8}{6720}$$

$$u_2 = \frac{x^6}{144} + \frac{\alpha x^7}{336} + \left(\frac{b}{2016} + \frac{\alpha^2}{2688}\right)x^8 + \left(\frac{5\alpha\beta}{36288} + \frac{59}{362880}\right)x^9$$
$$+\left(\frac{53\alpha}{1209600} + \frac{\beta^2}{72576}\right)x^{10} + \frac{359\beta x^{11}}{39916800} + \frac{359 x^{12}}{239500800}$$

$$u_3 = -\frac{x^7}{560} - \frac{3\alpha}{2240}x^8 + \left(-\frac{\beta}{4032} - \frac{\alpha^2}{2688}\right)x^9$$
$$+\left(-\frac{\alpha^3}{26880} - \frac{\alpha\beta}{6720} - \frac{5}{48384}\right)x^{10}$$
$$+\left(-\frac{\alpha^2\beta}{42240} - \frac{\beta^2}{63360} - \frac{157\alpha}{2661120}\right)x^{11}$$
$$+\left(-\frac{101\beta}{7983360} - \frac{\alpha\beta^2}{190080} - \frac{283\alpha^2}{31933440}\right)x^{12}$$
$$+\left(-\frac{821\alpha\beta}{207567360} - \frac{5189}{2075673600} - \frac{\beta^3}{2471040}\right)x^{13}$$
$$+\left(-\frac{49\alpha}{65894400} - \frac{265\beta^2}{581188608}\right)x^{14}$$
$$-\frac{74609\beta x^{15}}{435891456000} - \frac{74609 x^{16}}{3487131648000}$$

$$u_4 = \frac{769 x^{11}}{39916800} + \frac{307\alpha x^{12}}{21772800} + \left(\frac{17\beta}{5930496} + \frac{\alpha^2}{299520}\right)x^{13}$$
$$+\left(\frac{\alpha^3}{4193280} + \frac{18143}{17435658240} + \frac{5903\alpha\beta}{4358914560}\right)x^{14}$$
$$+\left(\frac{18043\alpha^2\beta}{13767436800} + \frac{2269\beta^2}{16345929600} + \frac{125603\alpha}{261534873600}\right)x^{15}$$
$$+\left(\frac{20569\beta}{209227898880} + \frac{293\alpha^2}{6199345152} + \frac{27119\alpha\beta^2}{1046139494400}\right)x^{16}$$
$$+\left(\frac{88631\alpha\beta}{5081248972800} + \frac{884413}{50812489728000} + \frac{27119\beta^3}{17784371404800}\right)x^{17}$$
$$+\left(\frac{2657033\alpha}{914624815104000} + \frac{189169\beta^2}{128047474114560}\right)x^{18}$$
$$+\frac{56433031\beta x^{19}}{121645100408832000} + \frac{56433031 x^{20}}{1216451004088320000}$$

Then, the approximate solution of Eq. (21) with boundary conditions is given by;

$u = u_0 + u_1 + u_2 + ...$

By the boundary conditions $u(1) = 0$, $u'(1) = 1$ and Maple Program for $N = 3$, we have;

$\alpha = -5.840342206, \quad \beta = 11.06223243$

For $N = 4$, we have

$\alpha = -5.840370798, \quad \beta = 11.06246972$

For $N = 5$, we have,

$\alpha = -5.840369988, \quad \beta = 11.06246677$

Table 2: Numerical results for example 2

x	Error of ST-HPM(N=3) $u_0 + u_1 + u_2$	Error of ST-HPM (N=4) $u_0 + u_1 + u_2 + u_3$	Error of ST-HPM (N=5) $u_0 + u_1 + u_2 + u_3 + u_4$
0	0	0	0
0.1	5.560824 e - 4	1.19158 e - 5	1.11231 e - 5
0.2	1.355568 e - 3	6.35276 e - 5	6.30883 e - 5
0.3	9.932615 e - 4	4.30479 e - 5	4.02078 e - 5
0.4	2.855579 e - 4	1.12510 e - 5	5.11870 e - 6
0.5	9.598820 e - 5	5.40440 e - 6	1.07300 e - 7
0.6	1.309530 e - 4	9.16100 e - 6	2.55500 e - 6
0.7	1.137292 e - 3	6.12610 e - 5	2.95450 e - 5
0.8	2.039742 e - 3	1.35428 e - 4	5.47810 e - 5
0.9	1.144457 e - 3	1.17183 e - 4	1.66550 e - 5
1	1.711132 e - 3	2.12412 e - 4	2.02160 e - 5

In Table 2, we list the numerical results were obtained by proposed method (ST-HPM) at $x = 0\ (0.1)1$. As can be seen from Table 2, the error decreased when the integer N is increased until $N = 5$.

The curve in Fig. 5 shows an approximate solution for example 2.

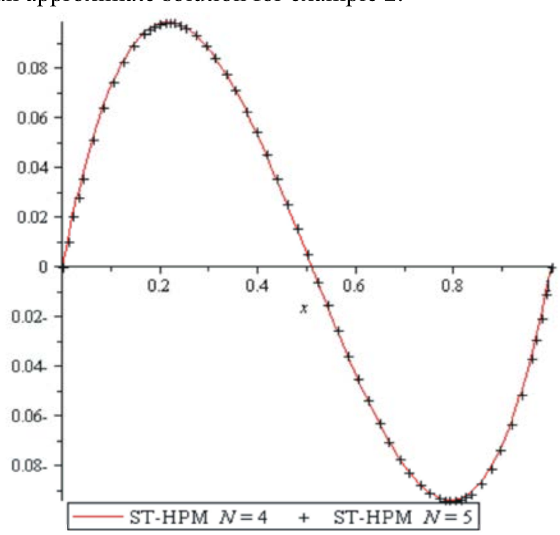

Fig. 5: The approximate solution by using (ST-HPM) for example 2

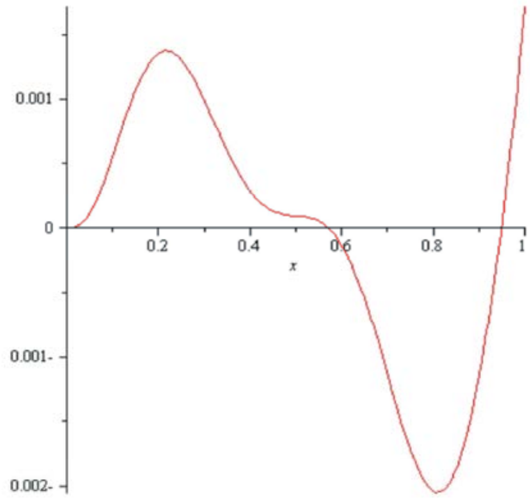

Fig. 6: It illustrates the error for $N = 3$

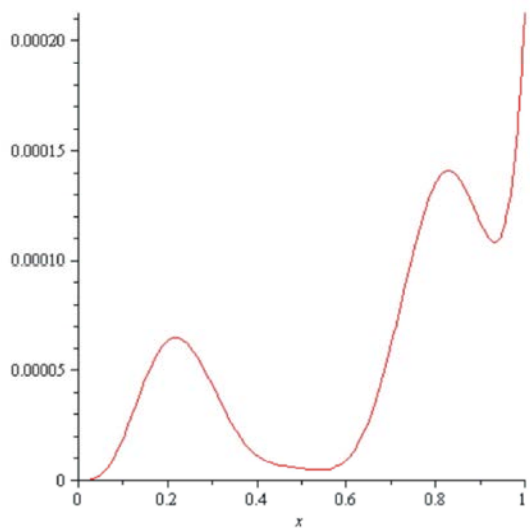

Fig. 7: It illustrates the error for $N = 4$

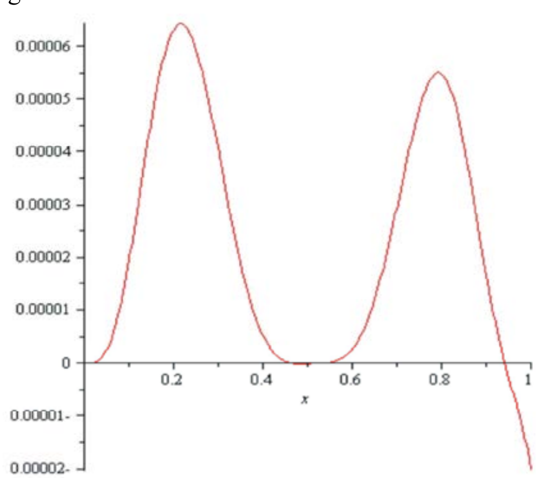

Fig. 8: It illustrates the error for $N = 5$

CONCLUSION

In this work, we have proposed a modification of the homotopy perturbation method through combining it with a sumuda transform. The aim of this approach is to obtain approximate solutions that are much closed to the exact solutions. The efficiency and accuracy of the presented hybrid method (ST-HPM) are validated through some examples. The results show that (ST-HPM) is a powerful and good technique for obtaining exact or approximate solutions of many systems of nonlinear differential equations. The computations associated in this work are performed by using the Maple software.

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