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Analytic Semi group With Partial Differential Equations in A Bannach Space

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Abstract: The question of generating analytic semigroup and existence or non-existence of solution to a given partial differential equation under some non-linear conditions always comes to mind. This paper considers a fractional heat equation or partial differential equation in Banach space and uses it to establish a fundamental solution of the homogeneous part of a given integro-differential equation. It further shows that the fundamental solution generates an analytic semi-group. In addition its considers existence of a mild solution of a given heat equation that do not have a global existence for all time t. Finally, we establish that the mild solution satisfies mean time continuity.

Key words: Partial differential equation · Fundamental solution · Analytic semigroup · Mild solution and Mean time continuity

INTRODUCTION

Consider the fractional heat equation or partial differential equation in Banach space.

$$\frac{du(t)}{dt} = -k(-\Delta)^{\frac{\alpha}{2}}u + m(t,u(x,t)), \quad t>0 \quad \text{and}$$

$$m(t,u(x,t)) = g(t,u(x,t)) + k(u)(t)$$

$$\frac{du(t)}{dt} = -k(-\Delta)^{\frac{\alpha}{2}}u(x,t) + g(t,u(x,t)) + k(u)(t),\ t>t_0$$
$$u(0,x) = u_0(x) \qquad (1)$$

where, $k(u)(t) = \int_{t_0}^{t} a(t-s)g(s,u(s))ds$

$u_0 : \Re \to \Re$ is bounded and measurable function, $\alpha \in (0,2),\ k>0,\ \Delta^{\frac{\alpha}{2}}$ is the laplace generator of analytic semigroup T_t, $t \geq 0$ on $T_t = e^{\frac{-\|x\|}{4t}}$

All measurable function $u_0 : \Re \to \Re_+,\ E[f(X_t)] = \int_{\Re} P(t,x) f(y)dy$

The above equation are discontinuous analogous of the equation introduced in [1]. Our solution to equation (1) is weak-predictable random field solution to the class of stochastic heat equation. We assume that the homogenous part of given stochastic heat equation to be $u_t - \Delta u = 0$, with this assumption we establish that there exist a fundamental solution to this homogenous equation and also shows that the fundamental solution generates an analytic semi-group. In [2], the author transformed mild solution to partial differential equation. Furthermore, if the mild solution satisfy some nonlinear initial condition, then they cease to exist mild solution at some finite time as in [3].

This paper was motivated by the work initiated by [4], where, they studied the large time behavior of the stochastic heat equation. The levy $N(dt, dx)$ has better modelling characteristics and performance of those natural phenomena of some real world modelling event (for example memory effect) unlike Brownian motion that has many imperfections. The levy noise $N(dt, dx)$ has a very rich and vast application in Finance and Economics.

Definition 1: We say that a process $\{u(t,x)\}_{x\in\Re, t>0}$ is a mild solution to equation (1) if the following is satisfied.

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$$u(t,x)=\int_{\Re}P(t,x)u_0(y)dy+\int_0^t\int_{\Re}P(t-s,x)\sigma(u(s),y)dyds \quad (2)$$

where $P(t,.)$ is the heat kernel, see [1]. If in addition $\{u(t,x)\}_{x\in\Re,t>0}$ satisfies the following condition $\sup_{0\le t\le T}\sup_{x\in\Re}E|u(t,x)^2|<\infty$ for all $T > 0$, then we say that $\{u(t,x)\}_{x\in\Re,t>0}$ is a random field solution to equation **(1)**.

If we let $\gamma(\beta)=\frac{1}{2\pi}\int_{\Re}\frac{d\xi}{\beta+2\mathrm{Re}\psi(\xi)}$ for all $\beta > 0$, ψ is the exponent of the Levy process with the requirement that $\gamma(\beta) < \infty$.

Definition 2: Mean Time Continuity: The mild solution equation (2) is said to be mean continuous in time if $t_1 < t_2$ and $\lim_{\delta\downarrow 0}\sup_{|t_1-t_2|<\delta}E|u(t_2,x)-u(t_1,x)|=0$ for a fixed $x\in\Re^d$.

MATERIALS AND METHODS

Here we state some lemmas and assumptions used in this paper

Assumption 1: We assume that initial condition is non-negative real value function $u_0:\Re\to\Re$ and $g:\Re\to\Re$ is globally lipschitz function satisfying $g(x)\le L_\sigma|x|$. That is, the function σ satisfies $\sigma(x)\le L_\sigma|x|$ with L_σ being positive number.

Assumption 2: We assume there exist positive function J and finite positive constant, Lip_σ such that for all $x, y, h \in \Re$, $|\sigma(0, h)| \le J(h)$ and $|\sigma(x,h)-\sigma(y,h)|\le J(h)Lip_\sigma|x-y|$. Function J is assumed to satisfy $\int_{\Re}J(h)\vee(dh)\le k$, where k is some finite positive constant and d is the dimension.

Assumption 3: We assume that k - 1 for simplicity.

Preposition1: Let $P(t, x)$ be the transition density of a strictly α stable process, if $P(t, 0) \le 1$ and $\alpha\ge 2$, then $P(t,\frac{1}{a}(x-y))\ge P(t,y)\ \forall x,y\in\Re^d$.

Theorem 1: Suppose assumption 1 holds, then for each $x \in \Re$, the unique solution to equation (1) is mean continuous in time t. That is, for each $x\in\Re^d$, the function $t\to E[|U(t,x)|]$ is continuous. This theorem require that we take the first moment of the mild solution and we therefore make use of the properties of heat kernel, applying some explicit bound the fractional kernel to obtain precise result.

Theorem 2: Suppose $C_{d,\alpha,\beta}<\frac{1}{kLip\sigma}$ for positive constant k and Lip_σ, then there exist a solution μ that is unique.

The proof of this theorem is based on these two lemmas below

Lemma 1: Let $A^\alpha u(t,x)=\int_0^t\int_{\Re^d}P^\alpha(t-s,x)\sigma(u(s,y)N(dsdy)$, suppose that $\|u\|_{1,\beta}<\infty$ for all $\beta > 0$ and $\sigma(u, h)$ satisfies the assumption 2, then $\|A^\alpha u\|_{1,\beta}\le C_{d,\alpha.\beta}k[1+Lip_\sigma\|u\|_{1,\beta}$

where $C_{d,\alpha,\beta}:=\frac{2C(d,\alpha)d+\alpha}{d+\alpha-1}\frac{\Gamma(\gamma+1)}{\beta^{\gamma+1}}$

Proof: Taking first moment of the solution. We have,

$$E|A^\alpha u(t,x)|=\int_0^t\int_{\Re^d}|P^\alpha(t-s,x)|E|\sigma(u(s),h)|\vee(dh)ds$$

$$\le k\int_0^t\int_{\Re^d}|P^\alpha(t-s,x)|[1+Lip_\sigma E|u(s)dsdx$$

Multiply through by exponent $(-\beta t)$, to get;

$$e^{-\beta t}E|A^\alpha u(t,x)|\le k\int_0^t\int_{\Re^d}e^{-\beta(t-s)}|P^\alpha(t-s,x)|e^{-\beta t}[1+Lip_\sigma E|u(s)|]dsdx$$

$$\le kLip_\sigma\sup_{t\ge 0}\sup_{x\in\Re^d}(e^{-\beta t}[1+Lip_\sigma E|u(s)|]]\int_0^t\int_{\Re^d}e^{-\beta(t-s)}|P^\alpha(t-s,x)|dsdx$$

Taking the norm, we obtain that;

$$\| A^{\alpha}u \|_{1,\beta} \leq k[1 + Lip_{\sigma} \| u \|_{1,\beta}] \sup_{t\geq 0} \int_0^t \int_{\Re^d} e^{-\beta(t-s)} | P^{\alpha}(t-s,x) | dxds$$

$$\leq k[1 + Lip_{\sigma} \| u \|_{1,\beta}] \int_0^{\infty} \int_{\Re^d} e^{-\beta t} | P^{\alpha}(s,x) | dsdx$$

$$\leq k[1 + Lip_{\sigma} \| u \|_{1,\beta}] \int_0^{\infty} \int_{\Re^d} e^{-\beta s} \{C(\frac{s}{| x |^{d+\alpha}} \wedge s^{-\frac{d}{\alpha}})\} dsdx.$$

Let us assume that $\frac{s}{| x |^{d+\alpha}} \leq s^{-\frac{d}{\alpha}}$ which holds only when $| x |^{\alpha} \geq s$. Then,

$$\| A^{\alpha}u \|_{1,\beta} \leq C(d,\alpha)k[1 + Lip_{\sigma} \| u \|_{1,\beta}] \int_0^{\infty} e^{-\beta t} ds [s \int_{|x|\geq s} \frac{1}{\alpha} \frac{dx}{| x |^{d+\alpha}} + s^{-\frac{d}{\alpha}} \int_{|x|\leq s} \frac{1}{\alpha} dx]$$

$$= C(d,\alpha)k[1 + Lip_{\sigma} \| u \|_{1,\beta}] \int_0^{\infty} e^{-\beta t} ds [s(-\int_{-\infty}^{s^{\frac{1}{\alpha}}} x^{-(d+\alpha)} dx) + \int_{s^{\frac{1}{\alpha}}}^{\infty} x^{-(d+\alpha)} + 2s^{\frac{1-d}{\alpha}}]$$

$$= C(d,\alpha)k[1 + Lip_{\sigma} \| u \|_{1,\beta}] \int_0^{\infty} e^{-\beta s} ds [s(-\frac{y^{-(d+\alpha-1)}}{1-d-\alpha}\Big|_{-\infty}^{s^{\frac{1}{\alpha}}} + \frac{y^{-(d+\alpha-1)}}{1-d-\alpha}\Big|_{s^{\frac{1}{\alpha}}}^{\infty}) + 2s^{\frac{1-d}{\alpha}}]$$

$$= C(d,\alpha)k[1 + Lip_{\sigma} \| u \|_{1,\beta}] \int_0^{\infty} e^{-\beta s} ds [\frac{2}{d+\alpha-1} s^{\frac{1+(1-d-\alpha}{\alpha}} + 2s^{\frac{1-d}{\alpha}}$$

Thus, $\| A^{\alpha}u \|_{1,\beta} \leq 2C(d,\alpha)k[1 + Lip_{\sigma} \| u \|_{1,\beta}] \frac{d+\alpha}{d+\alpha-1} \int_0^{\infty} s^{\gamma} e^{-\beta s} ds$, where $\gamma = \frac{1-d}{\alpha}$

Therefore, $\| A^{\alpha}u \|_{1,\beta} \leq 2C(d,\alpha)k[1 + Lip_{\sigma} \| u \|_{1,\beta}][\frac{d+\alpha}{d+\alpha-1} \frac{\Gamma(\gamma+1)}{\beta^{\gamma+1}}]$ (3)

If u and v are two predictable random field solutions, then by equation (3) and theorem 2 we have,

$\| u \|_{1,\beta} + \| v \|_{1,\beta} \leq C_{d,\alpha,\beta}[kLip_{\sigma} \| u - v \|_{1,\beta}$.

By assumption made earlier that, U be an open subset of $(0, \infty) \times X_{\alpha}$. A function f is said to satisfy assumption F mentioned earlier if for every $(t, x) \in U$ there exist a neighborhood $V \subset U$ of (t, x) and constant $L > 0$, $0 < \theta < 1$ such that $\| f(s_1,u) - f(s_2,v) \| \leq L \| s_1 - s_2 \|^{\theta} + \| u - v \|_{\alpha}$ for all (s_1, u) and (s_2, u) in V, which implies a continuous function u defined from U into X satisfying $u(t) = S(t-t_0)u_0 + \int_{t_0}^{t} S(t-s)f(s,u(s) + k(u)sds,\ t \in U$ is a mild solution.

Hence, there exists a mild solution to (1). Omaba (2016)

Lemma 2: For, $\beta > 0, 0 < t_1 < t_2, x \in \Re^d$. Then,

$$\int_{\Re^d} [P(t_2,x) - P(t_1,x)]u(y)dy | \leq 2C_0(d,\alpha) \frac{d+\alpha}{d+\alpha-1} (t_2^{(\frac{1-d}{\alpha})} - t_1^{(\frac{1-d}{\alpha})})$$

Proof: Let $\int_{\Re^d} [P(t_2,x) - P(t_1,x)]u(y)dy = D_1$. Then,

$$|D_1| = |\int_{\Re^d} [P(t_2,x) - P(t_1,x)]u(y)dy| \le \sup_{x\in\Re^d} |u(0,y)| \int_{\Re^d} P[(t_2,x) - P(t_1,x)]dx$$
$$= C_0 \int_{\Re^d} |P(t_2,x) - P(t_1,x)|\, dx$$

RESULTS

Establishing a fundamental solution of the Homogeneous part of the Integro-Differential Equation Given,

$$\frac{du(t)}{dt} = -k(-\Delta)^{\frac{\alpha}{2}} u(x,t) + g(t,u(x,t)), \ \ t > t_0 \qquad (4)$$
$$u(0,x) = u_0(x)$$

when the most right of the equation is equal to zero, $\alpha = 2$, $k = 1$, our stochastic heat equation becomes homogeneous equation. That is,

$u_t - \Delta u = 0, t > 0, \underline{x} \in \Re^n$ with initial condition $u(\underline{x},0) = f(x)$

We find our u_t, Δu and substitute to our homogeneous equation.
We start by seeking a solution with the special structure

$$u(\underline{x},t) = \frac{1}{t^\alpha} v(\frac{r}{t^\beta}) \qquad (5)$$

where $r = \|\underline{x}\| = \sqrt{x_1{}^2 + x_2{}^2 + ... + x_n{}^2}$, so that v, is radially symmetric and α,β are constants to be determined.

The formula for Laplacian of radially symmetric function is given as;

$$\Delta v(r) = v''(r) + \frac{n-1}{r} v'(r)$$

Recall special solution $u(\underline{x},t) = \frac{1}{t^\alpha} v(\frac{r}{t^\beta})$ and define $\Delta u = \frac{1}{t^\alpha} \Delta v(\frac{r}{t^\beta})$

Differentiate equation (5) with respect to t gives,

$$u_t(\underline{x},t) = \frac{1}{t^\alpha}\frac{\partial}{\partial t} v(\frac{r}{t^\beta}) + \frac{\partial}{\partial t}(\frac{1}{t^\alpha}) v(\frac{r}{t^\beta}) = t^{-\alpha} v'(\frac{r}{t^\beta})(-\beta r t^{-\beta+1}) - \alpha t^{-\alpha-1} v(\frac{r}{t^\beta})$$
$$= -t^{-\alpha} v'(\frac{r}{t^\beta})\beta r t^{-(\beta+1)} - \alpha t^{-(\alpha+1)} v(\frac{r}{t^\beta}) = -t^{-\alpha} v'(\frac{r}{t^\beta})\beta r.\frac{t^\beta}{t^\beta} t^{-\beta+1} - \alpha t^{-(\alpha+1)} v(\frac{r}{t^\beta}) \qquad (6)$$

Let $z = \frac{r}{t^\beta}$ we have,

$$u_t = -\beta t^{-(\alpha+1)} v'(z) z - \alpha t^{-(\alpha+1)}(z) \qquad (7)$$

$$\Delta u = 1/t^{\alpha}\Delta v(\frac{r}{t^{\beta}}) \text{ and } \Delta v(r) = V''(r) + \frac{n-1}{r}V'(r) \tag{8}$$

Differentiate $\Delta v(\frac{r}{t^{\beta}})$ twice with respect to *r*.

We have,

$$\Delta u = 1/t^{\alpha}[v''(r) + \frac{n-1}{r}v'(r)] = t^{-\alpha}[\frac{1}{t^{2\beta}}v''(\frac{r}{t^{\beta}}) + \frac{n-1}{r}.\frac{1}{t^{\beta}}v'(\frac{r}{t^{\beta}})] \tag{9}$$

$$= t^{-(\alpha+2\beta)}v''(\frac{r}{t^{\beta}}) + \frac{n-1}{r\frac{t^{\beta}}{t^{\beta}}}t^{-(\alpha+\beta)}v'(\frac{r}{t^{\beta}})$$

Let $z = \frac{r}{t^{\beta}}$ we have,

$$\Delta u = t^{-(\alpha+2\beta)}v''(z) + \frac{n-1}{z}t^{-(\alpha+2\beta)}v'(z) \tag{10}$$

Substitute (7) and (10) into general homogeneous heat equation i.e $U_t - \Delta u = 0$, we have

$$-\beta t^{(\alpha+1)}zv'(z) - \alpha t^{-(\varepsilon+1)}v(z) - t^{-(\alpha+2\beta)}v''(z)\frac{n-1}{z}t^{-(\alpha+2\beta)}v'(z)$$

$$\Rightarrow \quad t^{-(\alpha+1)}[\beta zv'(z) + \alpha v(z)] + t^{-(\alpha+2\beta)}[v''(z) + \frac{n-1}{z}v'(z)] = 0 \tag{11}$$

Choose $\beta = 1/2$

$$\Rightarrow \quad \frac{z}{2}v'(z) + \alpha v(z) + v''(z) + \frac{n-1}{z}v'(z) = 0 \tag{12}$$

Choose $\alpha = n/2$

$$\Rightarrow \quad \frac{z}{2}v'(z) + \frac{n}{2}v(z) + v''(z) + \frac{n-1}{z}v'(z) = 0 \tag{13}$$

Multiply through by z^{n-1}, we have

$$\frac{z^n}{2}v'(z) + \frac{n}{2}z^{n-1}v(z) + z^{n-1}v''(z) + (n-1)z^{n-2}V'(z) = 0$$

$$\Rightarrow \quad \frac{1}{2}(z^{n-1}) + (z^{n-1}v') = 0 \tag{14}$$

For these derivatives to be equal zero, it implies the functions are constant.

That is, $\frac{1}{2}z^n v + z^{n-1}v' =$ constant

This is possible if and only if $\lim_{t\to\infty} v(z)=0$ and $\lim_{z\to\infty} v(z)=0$

$$\Rightarrow \quad z^{n-1}[\frac{z}{2}v+v'=0$$

$$z^{n-1}\neq 0 \Rightarrow \frac{z}{2}v+v'=0$$

$$\Rightarrow \quad v'/v=-z/2,$$

Integrate both sides, that is,

$$\int v'/vdz=\frac{-1}{2}\int zdz \Rightarrow Inv=-\frac{z^2}{4}+C \Rightarrow v=\ell^{-\frac{z^2}{4}}=A\ell^{-\frac{z^2}{4}}$$

Recall, $z=r/t^{\beta}=r/t^{1/2}, \beta=1/2$

Therefore, $v(\frac{r}{t^{\frac{1}{2}}})=Ae^{\frac{-r^2}{4t}}=Ae^{-\frac{|x\}}{4t}}$, since $r=|x|$

$$\Rightarrow \quad U(x,t)=\frac{1}{t^{\alpha}}.v(\frac{|x|}{t^{\frac{1}{2}}})=1/t^{\frac{n}{2}}.Ae^{-\frac{|x|}{4t}}$$

Hence, $$U(x,t)=At^{-\frac{n}{2}}e^{-\frac{|x|}{4t}} \tag{15}$$

Here our semi-group is $t^{-\frac{n}{2}}e^{-\frac{|x|}{4t}}$, where the generator of the semigroup is *A*.

To find *A*, we normalized *u* such that

$$\int_{\Re^n} u(x,t)dx=1 \Rightarrow \int_{\Re^n} At^{-\frac{n}{2}}e^{-\frac{|x|}{4t}}dx=1 \tag{16}$$

$$\Rightarrow At^{-\frac{n}{2}}\int_{\Re^n} e^{-\frac{|x|}{4t}}dx=1$$

$$=\int_{\Re}\int_{\Re} e^{-(x^2+y^2)}dxdy \tag{17}$$

We integrate using polar coordinate, letting $x=r\cos\theta, y=r\sin\theta, \Rightarrow x^2+y^2=r^2$

By Jacobian,

$dxdy = rdrd\ \theta$

Let change θ: 0 to 2π, *r*: o to ∞ and let $a=r^2 \Rightarrow da/dr=2r, dr=da/2r$

Equation (16) becomes $\frac{1}{2}\int_0^{2\pi} d\theta \int_0^{\infty} \ell^{-a} da = \pi$

We have been able to show that: $\int_{\Re} e^{-y^2} dy = \sqrt{\pi}$

Now, from normalization $(\sqrt{\pi})^n = 1$

$$\Rightarrow A.2^n(\pi^{\frac{1}{2}})^n = 1 \Rightarrow A.2^n.\pi^{\frac{1}{2}} = 1$$

We can make 2 and π have the same power, that is,

$$A.4^n.\pi^{\frac{n}{2}} = 1 \Rightarrow A(4\pi)^{\frac{n}{2}} = 1$$

Therefore, $A = (\frac{1}{4\pi})^{\frac{n}{2}}$ (18)

Substitute the value of A into equation (15). That is, $u(x,t) = At^{-\frac{n}{2}}e^{\frac{-|x|}{4t}}$

Where $t^{-\frac{n}{2}}e^{\frac{-|x|}{4t}}$ is the semigroup generated by A.

Hence

$$u(x,t) = (\frac{1}{4\pi})^{\frac{n}{2}} t^{-\frac{n}{2}} e^{-\frac{|x|}{4t}}$$

is the solution of heat equation $u_t - \Delta u = g, t > 0, x \in \Re$ and it is called Fundamental solution of homogenous heat equation.

Generation of Analytic Semi-group from the Fundamental Solution: We show that the fundamental solution generates an analytic semigroup

Let $T = e^{\frac{-|x|^2}{4t}} \Rightarrow Tt^{\frac{-n}{2}} = e^{\frac{-|x|^2}{4t}} t^{\frac{-n}{2}}$ (19)

When $\boldsymbol{n}$ = 0 we have,

$$Tt = e^{\frac{-|x|^2}{4t}} t$$

but when $n \geq 1$ we multiply both sides of equation (18) by $t^{\frac{n}{2}}$ to have

$$Tt = e^{\frac{-|x|^2}{4t}} t \text{ .}$$

Now, $Tt = e^{\frac{-|x|^2}{4t}} t$, at t = 0, We have $T(0)=e^{0}(0)=I \Rightarrow T(0)x=I(x)$ that is identity of x, first property of a semigroup.

Since, $Tt=e^{\frac{-|x|^2}{4t}}t$, $Ts=e^{\frac{-|x|^2}{4t}}s$ (20)

$$\Rightarrow Tt.Ts=e^{\frac{-|x|2}{4t}}t.e^{\frac{-|x|^2}{4s}}s=T(s+t).$$

Hence, $t^{-\frac{n}{2}}e^{\frac{-|x|}{4t}}$ is an analytic semigroup generated by fundamental solution of homogenous heat equation.

Existence of Mild Solution: As it was established earlier in this paper that the Laplacian generator of alpha process $(\Delta^{\frac{\alpha}{2}}u)$ generate an analytic semigroup $(T_t)_{t\geq 0}$ and $(\Delta^{\frac{\alpha}{2}}u)$ is invertible.The result of below shows that the mild solution to equation **(1)** is unique.

Theorem 2: Suppose the operator $\Delta^{\frac{\alpha}{2}}$ generates the analytic semigroup $T_t, t\geq 0$ with $\|T(t)\|\leq M, t\geq 0$ and $0\in\rho(\Delta)^{\frac{\alpha}{2}}$. If the maps f and g satisfy assumption 1 and the real valued map α is integrable on J, then equation **(1)** has a unique mild solution for every $u_0 \in X_0$.

Proof: To establish the existence of the mild solution we assume that, $0< T < \infty$. We now fix a point (t_0, u_0) in the open subset U of $[0,\infty)\times X_\alpha$ and choose $t_1' > t_0$ and $\delta > 0$ such that $\| f(s,u)-f(s,v)\leq L_0\|u-v\|_\alpha$ for all (s, u) and (s, v) in V, with some constant $L_0>0$ hold for function f and g on the set.

$$V=\{(t,x)\in U: t_0\leq t_1', \|x-u_0\|_\alpha \leq \delta \tag{21}$$

Let $B_1=\sup_{t_0\leq t\leq t_1'}\|f(t,u_0)\|$ and $B_2=\sup_{t_0\leq t\leq t_1'}\|g(t,u_0)\|$

Choose $t_1'>t_0$ such that $\|T(t-t_0)-I\|\leq\|A^{\alpha}U_0\leq\frac{1}{2}\delta$ (22)

For all $t_1'>t_0$ and $t_1'-t_0<\min\{t_1^1-t_0[\frac{\delta}{2}C_\alpha^{-1}(1-\alpha)\{(L_0\delta+B_1)+a_T(L_0\delta+B_2)\}^{-1}]^{\frac{1}{1-\alpha}}\}$ (23)

where C_a is the positive constant depending on α satisfying

$$\|A^{\alpha}T(t)\leq C_\alpha t^{-\alpha} \quad \text{for } t>t_0 \tag{24}$$

where

$$a_T=\int_0^T|a(s)|\,ds. \tag{25}$$

Let Y $Y=C[(t_0,t_1);X]$ be endowed with supremum norm $\|y\|Y$.

Therefore, Y is a Banach Space.

We define map on Y by $Fy = \bar{y}$ and $\bar{y}$ is given as

$$\bar{y}(t) = T(t-t_0)A^{\alpha}u_0 + \int_{t_0}^{t} A^{\alpha}T(t-s)[f(s, A^{-\alpha}y(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau, A^{-\alpha}y(\tau))d\tau]ds.$$

Now, for every $y \in Y, Fy(t_0) = A^{\alpha}u_0$ and for $t_0 \le s \le t \le t_1$, we have

$$\begin{aligned} Fy(t) - Fy(s) &= [T(t-t_0) - T(s-t_0)]A^{\alpha}u_0 + \int_{s}^{t} A^{\alpha}T(t-\tau)\{f(\tau, A^{\alpha}y(\tau)) \\ &+ \int_{t_0}^{\tau}[a(\tau-\eta)g(\eta, A^{-\alpha}y(\eta))d\eta]d\tau + \int_{t_0}^{s} A^{\alpha}\{T(t-\tau) - T(s-\tau)\}\{f(\tau, A^{-\alpha}y(\tau) \\ &+ \int_{t_0}^{\tau} a(\tau-\eta)g(\eta, A^{-\alpha}y(\eta))d\eta\}d\tau \end{aligned} \tag{26}$$

By assumption (2) on the function f and g together with equation (24) and (25) we have the fact that $F{:}Y \to Y$

Theorem 3: Let S be nonempty closed and bounded set given by

$$S = \{y \in Y : y(t_0) = A^{\alpha}u_0, \| y(t) - A^{\alpha}u_0 \| \le \delta \tag{27}$$

Then, for $y \in S$, we have

$$\begin{aligned} \| Fy(t) - A^{\alpha}u_0 \| &\le \| T(t-t_0) - I \| \, \| A^{\alpha}u_0 \| \\ &+ \int_{t_0}^{t} \| A^{\alpha}T(t-s) \| \, \| f(s, A^{-\alpha}y(s)) - f(s,u_0 \| ds \\ &+ \int_{t_0}^{t} \| A^{\alpha}T(t-s) \| [\int_{t_0}^{s} | a(s-\tau) | \, \| g(\tau,u_0) \| d\tau]ds \\ &+ \int_{t_0}^{t} \| A^{\alpha}T(t-s) \| \, \| f(s,u_0) \| ds \\ &+ \int_{t_0}^{t} \| A^{\alpha}T(t-s) \| [\int_{t_0}^{t} | a(s-\tau) | \, \| g(\tau,u_0 \| d\tau]ds \\ &\le \frac{1}{2}\delta + C_{\alpha}(1-\alpha)^{-1}[(L_0\delta + B_1) + a_{\tau}(L_0\delta + B_2)](t_1 - t_0)^{1-\alpha} \\ &\le \delta \end{aligned} \tag{28}$$

By the last two inequalities and follows from equation (22) and (23), we have that, $F{:}\ S \to$ S.
Now, we show that F is a strict contraction on S.

Let y and z be in S; then

$$\| Fy(t) - Fz(t) \| = \| \bar{y}(t) - \bar{z}(t) \|$$
$$\leq \int_{t_0}^{t} \| A^{\alpha} T(t-s) \| \| f(s, A^{\alpha} y(s) - f(s, A^{\alpha} z(s) \| ds$$
$$+ \int_{t_0}^{t} \| A^{\alpha} T(t-s) \| [\int_{t_0}^{s} | a(s-\tau) | \| g(\tau, A^{\alpha} y(\tau)) - g(\tau, A^{-\alpha} z(\tau)) \| d\tau] ds \tag{29}$$

By our assumption (2) on function *f* and *g* and making use of equation (22)and (23), we get

$$\| Fy(t) - Fz(t) \| \leq L_0 [(1 + a_T) \int_{t_0}^{t} \| A^{\alpha} T(t-s) \| ds] \| y - z \|_Y$$
$$\leq L_0 (1 + a_T) C_{\alpha} (1-\alpha)^{-1} (t_1 - t_0)^{1-\alpha} \| y - z \|_Y$$
$$\leq \frac{1}{\delta} L_0 \delta (1 + a_T) C_{\alpha} (1-\alpha)^{-1} (t_1 - t_0)^{1-\alpha} \| y - z \|_Y$$
$$\leq \frac{1}{\delta} [L_0 \delta + B_1 + a_T (L_0 \delta + B_2)] C_{\alpha} (1-\alpha)^{-1} (t_1 - t_0)^{1-\alpha} \| y - z \|_Y$$
$$\leq \frac{1}{2} \| y - z \|_Y \tag{30}$$

By the last inequality and equation (23), thus *F* is a strict contraction map from *S* into *S* and by Banach contraction principle, there exist a unique fixed point *y* of *F*. That is, there exist $y \in S$ such that,

$$Fy = y = \bar{y} \tag{31}$$

Now, let $u = A^{-\alpha} y$. Then for $t \in [t_0, t_1]$, we have

$$u(t) = A^{-\alpha} y(t) = T(t - t_0) u_0 + \int_{t_0}^{t} T(t-s)[f(s, u(s)) + ku(s)] ds$$

Hence *u* is a unique mild solution to equation (21).

Mean Time Continuity of the Mild Solution: Establishing the mean continuity in time of the mild solution, by Duhame's principle the mild solution will be of the form;

$$u(t,x) = \int_{\Re^d} P(t,x) u_0(y) dy + \int_0^t \int_{\Re^d} P(t-s,x) \sigma(u(s,x)) ds dx \tag{32}$$

With *P*(*t*,.) be the heat kernel. We impose the following integrability condition on the mild solution. That is, $\sup_{t \geq 0} \sup_{x \in \Re^d} | u(t,x) | < \infty$, where $0 < t_1 < t_2 < t$

$$u(t_1,x) = \int_{\Re^d} P(t_1, x-y)u_0(y)dy + \int_0^{t_1}\int_{\Re^d} P(t_1 - s, x-y)\sigma(u(s,x))dsdx \tag{33}$$

$$u(t_2,x) = \int_{\Re^d} P(t_2, x-y)u_0(y)dy + \int_0^{t_2}\int_{\Re^d} P(t_2 - s, x-y)\sigma(u(s,x))dsdx \tag{34}$$

By integral property: $\int_a^c = \int_a^b + \int_b^c \Rightarrow \int_0^{t_2} = \int_0^{t_1} + \int_{t_1}^{t_2}$ which implies that,

$$u(t_2,x) = \int_{\Re^d} P(t_2, x-y)u_0(y)dy + \int_0^{t_1}\int_{\Re^d} P(t_2 - s, x-y)\sigma(u(s,y))dsdy + \int_{t_1}^{t_2}\int_{\Re^d} P(t_2 - s, x-y)\sigma(u(s,y))dsdy$$

Therefore, for a fixed, $x \in \Re^d$

$$\begin{aligned} u(t_2,x) - u(t_1,x) &= \int_{\Re^d} P(t_2, x-y)u_0(y)dy + \int_0^{t_1}\int_{\Re^d} P(t_2 - s, x-y)\sigma(s,y)dsdy + \int_{t_1}^{t_2}\int_{\Re^2} P(t_2 - s, x-y)\sigma(s,y)dyds - \\ &\quad (\int_{\Re^2} P(t_1, x-y)u_0(y)dy + \int_0^{t_1}\int_{\Re^d} P(t_1 - s, x-y)\sigma(s,y)dsdy)] \\ &= \int_{\Re^d} (P(t_2, x-y) - P(t_1, x-y))u_0(y)dy + \int_0^{t_1}\int_{\Re^d} P((t_2 - s, x-y) - P(t_1 - s, x-y))\sigma(s,y)dsdy \\ &\quad + \int_{t_1}^{t_2}\int_{\Re^d} P(t_2 - s, x-y)\sigma(s,y)dsdy. \end{aligned}$$

$$\begin{aligned} |u(t_2,x) - u(t_1,x)| &= |\int_{\Re^d} (P(t_2, x-y) - P(t_1, x-y))u_0(y)dy + \int_0^{t_1}\int_{\Re^d} P((t_2 - s, x-y) - P(t_1 - s, x-y))\sigma(s,y)dsdy \\ &\quad + \int_{t_1}^{t_2}\int_{\Re^d} P(t_2 - s, x-y)\sigma(s,y)dsdy| \end{aligned}$$

Taking the limit as t_1 approaches t_2, we have

$$\begin{aligned} \lim_{t_1 \uparrow t_2} |u(t_2,x) - u(t_1,x)| &= |\int_{\Re^d} P((t_2, x-y) - P(t_2, x-y))u_0(y)dy + \int_0^{t_2}\int_{\Re^d} P((t_2 - s, x-y) - P(t_2 - s, x-y)\sigma(s,y)dsdy \\ &\quad + \int_{t_2}^{t_2}\int_{\Re^d} P(t_2 - s, x-y)\sigma(s,y)dsdy| \\ &= |0 + 0 + 0| = 0. \end{aligned}$$

Using the estimate property on the heat kernel of alpha-stable process, we have

$$P(t_2, x-y) - P(t_1, x-y) = (t_2^{-\frac{d}{\alpha}} \wedge \frac{t_2}{|x-y|^{d+\alpha}}) - (t_1^{\frac{-d}{\alpha}} \wedge \frac{t_1}{|x-y|^{d+\alpha}}).$$

Therefore, $E|D_1| \le C_0\{\int_{\Re^d}(t_2^{-\frac{d}{\alpha}} \wedge \frac{t_2}{|x-y|^{d+\alpha}})dy - \int_{\Re^d}(t_1^{-\frac{d}{\alpha}} \wedge \frac{t_1}{|x-y|^{d+\alpha}})dy\}$

But $\int_{\Re^d}(t_2^{-\frac{d}{\alpha}} \wedge \frac{t_2}{|x-y|^{d+\alpha}})dy = t_2 \int_{|x-y|\ge t_2^{\frac{1}{\alpha}}} (\frac{dy}{|x-y|^{d+\alpha}}) + t^{\frac{-d}{\alpha}} \int_{|x-y|<t_2^{\frac{1}{\alpha}}} dy$

$$= \frac{2C(d,\alpha)}{d+\alpha-1} t_2^{\frac{1+(1-d-\alpha)}{\alpha}} + 2t_2^{\frac{(1-d)}{\alpha}} = 2C(d,\alpha)\frac{d+\alpha}{d+\alpha-1} t_2^{\frac{(1-d)}{\alpha}}.$$

Doing the same for the integral on t_1, we have

$$|D_1| \le 2C_0C(d+\alpha)\frac{d+\alpha}{d+\alpha-1}(t_2^{(\frac{1-d}{\alpha})} - t_1^{(\frac{1-d}{\alpha})}).$$

Combining theorem 1, lemma1and lemma 2, we have

$$E|u(t_2,x)-u(t_1,x)| \le 2C_0C(d,\alpha)\frac{d+\alpha}{d+\alpha-1}(t_2^{\frac{-d}{\alpha}} - t_1^{\frac{-d}{\alpha}}) + 2kLip_\sigma C(d,\alpha)\|u\|_{1,\beta}\frac{d+\alpha}{d+\alpha-1}$$
$$(-e^{\beta t_2}\int_{t_1}^{t_2} z^{\frac{1-d}{\alpha}dz} + e^{\beta t_1}\int_0^{t_1} z^{\frac{1-d}{\alpha}} e^{-\beta z}dz + 2kLip_\sigma C(d,\alpha)\|u\|_{1,\beta}\, e^{\beta t_2}\frac{d+\alpha}{d+\alpha-1}\int_0^{t_2-t_1} z^{\frac{1-d}{\alpha}} e^{-\beta t}dz$$

Then $\lim_{\delta\downarrow 0} \sup_{|t_1-t_2|<\delta} E|u(t_2,x)-u(t_1,x)| \le 0$ and therefore $\lim_{\delta\downarrow 0} \sup_{|t_1-t_2|<\delta} E|u(t_2,x)-u(t_1,x)| = 0$

Hence, $|u(t_2,x)-u(t_1,x)| = 0$. This implies that the mild solution to the given stochastic heat problem is mean continuous in time.

CONCLUSION

Herein, we have been able to establish a fundamental solution of the homogeneous part of the partial differential equation using fractional heat equation in Banach space. We also generate an analytic semi-group using the fundamental solution established from fractional heat equation. Furthermore, we established the existence of a mild solution from the stochastic heat equation that do not posses a global existence of solution for all time t and find out that our mild solution satisfy mean time continuity.

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