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Generalized Ulam-Hyers Stability of N-Dimensional Cubic Functional Equation in FNS and RNS: Various Methods

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Abstract: In this paper, the authors investigate the general solution in vector space and generalized Ulam-Hyers stability of *n*-dimensional cubic functional equation

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = (n-6) f\left(\sum_{j=1}^{n} x_j\right) + 4 \sum_{1\le i<j\le n} f(x_i + x_j) - \left(\frac{n-2}{2}\right)\sum_{j=1}^{n} f(2x_j)$$

where $x_{ij} = \begin{cases} -x_j & if \quad i = j \\ x_j & if \quad i \neq j \end{cases}$

and $n \neq 6$ is a positive integer using fuzzy normed space (FNS) and random normed space (RNS) by direct and fixed point methods.

Key words: Fuzzy normed space · Random normed spaces · Cubic functional equation · Ulam -Hyers stability · Fixed point method

INTRODUCTION

The stability of functional equations originated from a question of S.M. Ulam [1] concerning the stability of group homomorphisms. D.H. Hyers [2] gave a first confirmatory part respond to the difficulty of Ulam for Banach spaces. He proved the following celebrated theorem.

Theorem 1.1: [17] *Let X, Y be Banach spaces and let f :X → Y be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \tag{1.1}$$

for all *x, y* ∈ *X*. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all *x* ∈ *X* and *a* : *X* → *Y* is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \le \varepsilon \tag{1.3}$$

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for all $x \in X$. Moreover, if $f(tx)$ is continuous in $1 \in R$ for each fixed $x \in X$, then the function a is linear.

Hyers' theorem was generalized by T. Aoki [3] for additive mappings and by Th.M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias has provided a lot of power in the improvement of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Th.M. Rassias theorem was obtained by P. Gavruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. In 1982, J.M. Rassias [6] followed the innovative approach of the Th.M. Rassias theorem in which he replaced the factor

$\| x\|^p + \| y\|^p$ by $\| x\|^p \| y\|^q$ for with $p,q \in R$ $p+q \neq 1$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [7] by considering the summation of both the sum and the product of two p-norms in the sprit of Rassias approach. The stability problems of numerous functional equations have been expansively investigated by a number of authors and there are many attractive outcome concerning this problem (see [1, 11, 18, 21]). J.M. Rassias [8] first introduced and proved the Ulam stability of a cubic functional equation.

$$c(x+2y)+3c(x)=3c(x+y)+c(x-y)+6c(y). \qquad (1.4)$$

Also K.W. Jun and H.M. Kim [19] discussed the generalized Hyers-Ulam-Rassias stability of a cubic functional equation of the form

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \qquad (1.5)$$

During the last few decades, the stability problems of several cubic functional equations in various spaces such as Menger Probabilistic Normed Spaces, Random normed spaces and Non-Archimedean Fuzzy normed spaces, Banach spaces, orthogonal spaces etc. have been extensively investigated by a number of mathematicians (see [29, 30, 14, 41, 6, 8, 10, 31, 20, 42]). In this paper, the authors investigate the general solution and generalized Ulam-Hyers stability of n-dimensional cubic functional equation

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = (n-6) f\left(\sum_{j=1}^{n} x_j\right) + 4 \sum_{1\le i<j\le n} f(x_i+x_j) - \left(\frac{n-2}{2}\right)\sum_{j=1}^{n} f(2x_j) \qquad (1.6)$$

where $x_{ij} = \begin{cases} -x_j & if \quad i=j \\ x_j & if \quad i\neq j \end{cases}$

and $n \neq 6$ is a positive integer using fuzzy (FNS) and random normed spaces (RNS) by direct and fixed point methods.

General Solution of the Functional Equation (1.6): In this section, the authors present the general solution of the cubic functional equation (6). Throughout this section let us consider X and Y be real vector spaces.

Theorem 2.1: *If $f : X \to Y$ is a function satisfying the functional equation (1.6) for all* $x_1, x_2, x_3, \cdots, x_n \in X$ *then there exists a function $B : X^3 \to Y$ such that* $f(x) = B(x,x,x)$ *for all $x \in X$ where B is symmetric for each fixed one variable and additive for each fixed two variables.*

Theorem 2.2: *If the mapping $f : X \to Y$ satisfies the functional equation* (1.6) *for all* $x_1, x_2, x_3, \cdots, x_n \in X$ *then $f : X \to Y$ satisfying the functional equation* (1.5) for all *$x, y \in X$.*

ProofL: Let $f : X \to Y$ satisfies (1.6). Setting $(x_1, x_2, \cdots, x_n)$ by $(0,0,...,0)$ in (1.6), we get $f(0)=0$. Letting $(x_1, x_2, x_3, \cdots, x_n)$ by $(x,x,0,\cdots,0)$ in (1.6), we obtain

$$f(2x) = 2^3 f(x) \qquad (2.1)$$

for all $x \in X$. Using (2.1) in (1.6), we get

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = (n-6) f\left(\sum_{j=1}^{n} x_j\right) + 4 \sum_{1\le i<j\le n} f(x_i+x_j) - (4n-8)\sum_{j=1}^{n} f(x_j) \qquad (2.2)$$

for all $x_1, x_2, x_3, \cdots, x_n \in X$. Replacing $(x_1, x_2, \cdots, x_n)$ by $(x,0,\cdots,0)$ in (2.2), we have

$$f(-x) = -f(x) \tag{2.3}$$

for all $x \in X$. Again replacing $(x_1,x_2,x_3,x_4,\cdots,x_n)$ by $(x,x,x,0,\cdots,0)$ in (2.2), we arrive

$$f(3x)=3^3 f(x) \tag{2.4}$$

for all $x \in X$. In general for any positive integer m, $f(mx) = m^3 f(x)$.

Substituting $(x_1,x_2,x_3,x_4,\cdots,x_n)$ by $(x_2,x_1,x_1,0,\cdots,0)$ in (2.2), we arrive

$$3f(2x_1+x_2)+f(2x_1-x_2) = 8f(x_1+x_2)+24f(x_1)-6f(x_2) \tag{2.5}$$

for all $x_1,x_2 \in X$. Replacing x_2 by $-x_2$ in (2.5) and using oddness of f we get,

$$3f(2x_1-x_2)+f(2x_1+x_2) = 8f(x_1-x_2)+24f(x_1)+6f(x_2) \tag{2.6}$$

for all $x_1,x_2 \in X$. Adding (2.5) and (2.6), we arrive (1.5). By Theorem 2.1 [19] we derived our result.

Throughout this paper, we use the following notation for a given mapping $f : X \to Y$ such that

$$D\,f(x_1,x_2,\cdots,x_n) = \sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) - (n-6) f\left(\sum_{j=1}^{n} x_j\right) - 4\sum_{1\le i<j\le n} f(x_i+x_j) + \frac{(n-2)}{2}\sum_{j=1}^{n} f(2x_j)$$

for all $x_1,x_2,\ldots,x_n \in X$

Preliminaries of Fuzzy Normed Spaces: We use the definition of fuzzy normed spaces given in [7] and [24-27].

Definition 3.1: *Let X be a real linear space. A function* $N : X\times\mathbb{R}\to[0,1]$ *(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all x, y $\in$ X and all s, t, $\in \mathbb{R}$,*

($F1$) $N(x,c) = 0$ for $c \le 0$;

($F2$) $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;

($F3$) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;

($F4$) $N(x+y,s+t) \ge min\{N(x,s),N(y,t)\}$;

($F5$) $N(x;)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty} N(x,t) = 1$;

($F6$) for $x \neq 0$, $N(x;)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(X, t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 3.2: *Let* $(X,\|\cdot\|)$ *be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t>0,\ x\in X, \\ 0, & t\le 0,\ x\in X \end{cases}$$

is a fuzzy norm on X.

Definition 3.3: *Let (X, N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists x $\in$ X such that* $\lim_{n\to\infty} N(x_n - x,t) = 1$ *for all t > 0.* In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x.$

Definition 3.4: *A sequence x_n in X is called Cauchy if for each ε > 0 and each t > 0 there exists n_0 such that for all n ≥ n_0 and all p > , we have* $N(x_{n+p} - x_n,t) > 1-\varepsilon.$

Definition 3.5: *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 3.6: *A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$ then f is said to be continuous on X.*

Fuzzy Stability Results: Direct Method: Throughout this section, assume that $X(Z, N')$ and (Y, N) are linear space, fuzzy normed space and fuzzy Banach space, respectively.

Now, we investigate the generalized Ulam-Hyers stability of n-dimensional cubic functional equation (1.6).

Theorem 4.1: *Let $\beta \in \{-1,1\}$ be fixed and let $\alpha : X^n \to Z$ be a mapping such that for some d with* $0 < \left(\frac{d}{2^3}\right)^{\beta} < 1$

$$N'\left(\alpha\left(n^{\beta}x,n^{\beta}x,\cdots,n^{\beta}x,\right),r\right) \ge N'\left(d^{\beta}\alpha\left(x,x,\cdots,x\right),r\right) \tag{4.1}$$

for all $x \in X$ and all $r > 0$, $d > 0$ and $\lim_{k\to\infty} N'\left(\alpha\left(n^{\beta k}x_1, n^{\beta k}x_2, \cdots, n^{\beta k}x_n\right), n^{\beta 3k}r\right) = 1$ (4.2)

for all $x_1, x_2, \cdots, x_n \in X$ and all $r > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$N\left(Df(x_1, x_2, \cdots, x_n), r\right) \geq N'\left(\alpha(x_1, x_2, \cdots, x_n), r\right) \tag{4.3}$$

for all $r > 0$ and all $x_1, x_2, \cdots, x_n \in X$. Then the limit

$$C(x) = N - \lim_{k\to\infty} \frac{f(n^{\beta k}x)}{n^{\beta 3k}} \tag{4.4}$$

exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping such that

$$N\left(f(x) - C(x), r\right) \geq N'\left(\alpha(x, x, \cdots, x), r(n-6)\,|\, n^3 - d\,|\right) \tag{4.5}$$

for all $x \in X$ and all $r > 0$.

Proof: First assume $\beta = 1$. Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, x, \cdots, x)$ in (4.3), we get

$$N\left((n-6)f(nx) - n^3(n-2)f(x), r\right) \geq N'\left(\alpha(x, x, \cdots, x), r\right) \tag{4.6}$$

for all $x \in X$ and all $r > 0$. Replacing x by $n^k x$ in (4.6), we obtain

$$N\left(\frac{f(n^{k+1}x)}{n^3} - f(n^k x), \frac{r}{(n-6)n^3}\right) \geq N'\left(\alpha(n^k x, n^k x, \cdots, n^k x), r\right) \tag{4.7}$$

for all $x \in X$ and all $r > 0$. Using (4.1), ($F3$) in (4.7), we arrive

$$N\left(\frac{f(n^{k+1}x)}{n^3} - f(n^k x), \frac{r}{(n-6)n^3}\right) \geq N'\left(\alpha(x, x, \cdots, x), \frac{r}{d^k}\right) \tag{4.8}$$

for all $x \in X$ and all $r > 0$. It is easy to verify from (4.8), that

$$N\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, \frac{r}{(n-6)n^3 \cdot n^{3k}}\right) \geq N'\left(\alpha(x, x, \cdots, x), \frac{r}{d^k}\right) \tag{4.9}$$

holds for all $x \in X$ and all $r > 0$. Replacing r by $d^k r$ in (4.9), we get

$$N\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, \frac{d^k\, r}{(n-6)n^3 \cdot n^{3k}}\right) \geq N'\left(\alpha(x, x, \cdots, x), r\right) \tag{4.10}$$

for all $x \in X$ and all $r > 0$. It is easy to see that

$$\frac{f(n^k x)}{n^{3k}} - f(x) = \sum_{i=0}^{k-1}\left[\frac{f(n^{i+1}x)}{n^{3(i+1)}} - \frac{f(n^i x)}{n^{3i}}\right] \tag{4.11}$$

for all $x \in X$. From equations (4.10) and (4.11), we have

$$\begin{aligned} N\left(\frac{f(n^k x)}{n^{3k}} - f(x), \sum_{i=0}^{k-1}\frac{d^i\, r}{(n-6)n^3 \cdot n^{3i}}\right) &\geq min\bigcup_{i=0}^{k-1}\left\{\frac{f(n^{i+1}x)}{n^{3(i+1)}} - \frac{f(n^i x)}{n^{3i}}, \frac{d^i\, r}{(n-6)n^3 \cdot n^{3i}}\right\} \\ &\geq min\bigcup_{i=0}^{k-1}\left\{N'\left(\alpha(x,x,\cdots,x),r\right)\right\} \quad \geq N'\left(\alpha(x,x,\cdots,x),r\right) \end{aligned} \tag{4.12}$$

for all $x \in X$ and all $r > 0$. Replacing x by $n^m x$ in (4.12) and using (4.1), ($F3$), we obtain

$$\begin{aligned} N&\left(\frac{f(n^{k+m}x)}{n^{3(k+m)}} - \frac{f(n^m x)}{n^{3m}}, \sum_{i=0}^{k-1}\frac{d^i\, r}{(n-6)n^3 \cdot n^{3(i+m)}}\right) \\ &\geq N'\left(\alpha(x,x,\cdots,x), \frac{r}{d^m}\right) \end{aligned} \tag{4.13}$$

for all $x \in X$ and all $r > 0$ and all $m,k \geq 0$. Replacing r by $d^m x$ in (4.13), we get

$$\begin{aligned} N&\left(\frac{f(n^{k+m}x)}{n^{3(k+m)}} - \frac{f(n^m x)}{n^{3m}}, \sum_{i=m}^{m+k-1}\frac{d^i\, r}{(n-6)n^3 \cdot n^{3i}}\right) \\ &\geq N'\left(\alpha(x,x,\cdots,x),r\right) \end{aligned} \tag{4.14}$$

for all $x \in X$ and all $r > 0$ and all $m,k \geq 0$. Using ($F3$) in (4.14), we obtain

$$N\left(\frac{f(n^{k+m}x)}{n^{3(k+m)}} - \frac{f(n^m x)}{n^{3m}}, r\right) \geq N'\left(\alpha(x,x,\cdots,x), \frac{r}{\sum_{i=m}^{m+k-1}\frac{d^i}{(n-6)n^3 \cdot n^{3i}}}\right) \tag{4.15}$$

for all $x \in X$ and all $r > 0$ and all $m, k \geq 0$. Since $0 < d < n^3$ and $\sum_{i=0}^{k}\left(\frac{d}{n^3}\right)^i < \infty$, the cauchy criterion for convergence and ($F5$) implies that $\left\{\frac{f(n^k x)}{n^{3k}}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $C(x) \in Y$. So one can define the mapping $C : X \to Y$ by

$$C(x) = N - \lim_{k\to\infty}\frac{f(n^k x)}{n^{3k}}$$

for all $x \in X$ Letting $m = 0$ in (4.15), we get

$$N\left(\frac{f(n^k x)}{n^{3k}} - f(x), r\right) \geq N'\left(\alpha(x, x, \cdots, x), \frac{r}{\sum_{i=0}^{k-1} \frac{d^i}{(n-6)n^3 \cdot n^{3i}}}\right) \tag{4.16}$$

for all $x \in X$ and all $r > 0$. Letting $k \to \infty$ in (4.16) and using ($F6$), we arrive

$$N\big(f(x) - C(x), r\big) \geq N'\Big(\alpha(x, x, \cdots, x), (n-6)r(n^3 - d)\Big)$$

for all $x \in X$ and all $r > 0$. To prove C satisfies the (1.6), replacing $(x_1, x_2, \cdots, x_n)$ by $(n^k x_1, n^k x_2, \cdots, n^k x_n)$ in (4.3), respectively, we obtain

$$N\left(\frac{1}{n^{3k}} Df(n^k x_1, \cdots, n^k x_n), r\right) \geq N'\Big(\alpha(n^k x_1, \cdots, n^k x_n), n^{3k} r\Big) \tag{4.17}$$

for all $r > 0$ and all $x_1, \cdots, x_n \in X$. Now,

$$\begin{aligned}
&N\left(\sum_{i=1}^{n} C\left(\sum_{j=1}^{n} x_{ij}\right) - (n-6)C\left(\sum_{j=1}^{n} x_j\right) - 4\sum_{1 \leq i < j \leq n} C(x_i + x_j) + \frac{(n-2)}{2}\sum_{j=1}^{n} C(2x_j), r\right) \\
&\geq \min\left\{ N\left(\sum_{i=1}^{n} C\left(\sum_{j=1}^{n} x_{ij}\right) - \frac{1}{n^{3k}}\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} n^k x_{ij}\right), \frac{r}{5}\right),\right. \\
&N\left(-(n-6)C\left(\sum_{j=1}^{n} x_j\right) + \frac{(n-6)}{n^{3k}} f\left(\sum_{j=1}^{n} n^k x_j\right), \frac{r}{5}\right), \\
&N\left(-4\sum_{1 \leq i < j \leq n} C(x_i + x_j) + \frac{4}{n^{3k}}\sum_{1 \leq i < j \leq n} f(n^k(x_i + x_j)), \frac{r}{5}\right), N\left(\frac{(n-2)}{2}\sum_{j=1}^{n} C(2x_j) - \frac{1}{n^{3k}}\left(\frac{n-2}{2}\right)\sum_{j=1}^{n} f(n^k(2x_j)), \frac{r}{5}\right), \\
&\left.\left. -\frac{4}{n^{3k}}\sum_{1 \leq i < j \leq n} f(n^k(x_i + x_j)) + \frac{1}{n^{3k}}\frac{(n-2)}{2}\sum_{j=1}^{n} f(n^k(2x_j)), \frac{r}{5}\right)\right\}
\end{aligned} \tag{4.18}$$

for all $x_1, \cdots, x_n \in X$ and all $r > 0$. Using (4.17) and ($F5$) in (4.18), we arrive

$$\begin{aligned}
&N\left(\sum_{i=1}^{n} C\left(\sum_{j=1}^{n} x_{ij}\right) - (n-6)C\left(\sum_{j=1}^{n} x_j\right) - 4\sum_{1 \leq i < j \leq n} C(x_i + x_j) + \frac{(n-2)}{2}\sum_{j=1}^{n} C(2x_j), r\right) \geq \min\left\{1, 1, 1, 1, N'\Big(\alpha(n^k x_1, \cdots, n^k x_n), n^{3k} r\Big)\right\} \\
&\geq N'\Big(\alpha(n^k x_1, \cdots, n^k x_n), n^{3k} r\Big)
\end{aligned} \tag{4.19}$$

for all $x_1,\cdots,x_n \in X$ and all $r > 0$. Letting $k \to \infty$ in (4.19) and using (4.2), we see that

$$N\left(\sum_{i=1}^{n} C\left(\sum_{j=1}^{n} x_{ij}\right) - (n-6)C\left(\sum_{j=1}^{n} x_j\right) - 4\sum_{1\le i<j\le n} C(x_i + x_j) + \frac{(n-2)}{2}\sum_{j=1}^{n} C(2x_j), r\right) = 1 \tag{4.20}$$

for all $x_1,\cdots,x_n \in X$ and all $r > 0$. Using $(F2)$ in the above inequality gives

$$\sum_{i=1}^{n} C\left(\sum_{j=1}^{n} x_{ij}\right) = (n-6)C\left(\sum_{j=1}^{n} x_j\right) + 4\sum_{1\le i<j\le n} C(x_i + x_j) + \frac{(n-2)}{2}\sum_{j=1}^{n} C(2x_j)$$

for all $x_1,\cdots,x_n \in X$. Hence C satisfies the cubic functional equation (1.6). In order to prove $C(x)$ is unique, let $C'(x)$ be another cubic functional equation satisfying (1.6) and (4.5). Hence,

$$N(C(x)-C'(x),r) = N\left(\frac{C(n^k x)}{n^{3k}} - \frac{C'(n^k x)}{n^{3k}}, r\right) \ge min\left\{N\left(\frac{C(n^k x)}{n^{3k}} - \frac{f(n^k x)}{n^{3k}}, \frac{r}{2}\right), N\left(\frac{f(n^k x)}{n^{3k}} - \frac{C'(n^k x)}{n^{3k}}, \frac{r}{2}\right)\right\}$$
$$\ge N'\left(\alpha(n^k x, n^k x,\cdots,n^k x), \frac{(n-6)r\, n^{3k}(n^3-d)}{2}\right) \ge N'\left(\alpha(x,x,\cdots,x), \frac{(n-6)r\, n^{3k}(n^3-d)}{2d^k}\right)$$

for all $x \in X$ and all $r > 0$. Since

$$\lim_{k\to\infty} \frac{(n-6)r\, n^{3k}(n^3-d)}{2d^k} = \infty,$$

we obtain

$$\lim_{k\to\infty} N'\left(\alpha(x,x,\cdots,x), \frac{(n-6)r\, n^{3k}(n^3-d)}{2d^k}\right) = 1.$$

for all $x \in X$ and all $r > 0$. Thus

$$N(C(x)-C'(x),r) = 1$$

for all $x \in X$ and all $r > 0$, hence $C(x) = C'(x)$. Therefore $C(x)$ is unique.

For $\beta = -1$, we can prove the result by a similar method. This completes the proof of the theorem.

From Theorem 4.1, we obtain the following corollary concerning the generalized Ulam-Hyers stability for the functional equation (1.6).

Corollary 4.2: *Suppose that a function* $f: X \to Y$ *satisfies the inequality*

$$N\left(Df(x_1,x_2,\ldots,x_n),r\right) \geq \begin{cases} N'\left(\varepsilon,r\right), & \\ N'\left(\varepsilon\sum_{i=1}^{n}\|x_i\|^s,r\right), & s\neq 3; \\ N'\left(\varepsilon\prod_{i=1}^{n}\|x_i\|^s,r\right), & s\neq \frac{3}{n}; \\ N'\left(\varepsilon\left(\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right),r\right), & s\neq \frac{3}{n}; \end{cases}$$

for all $x_1,\cdots,x_n \in X$ and all $r > 0$, where ε, s are constants with $\varepsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N\left(f(x)-C(x),r\right) \geq \begin{cases} N'\left(\varepsilon,(n-6)\,|\,n^3-1\,|\,r\right), \\ N'\left(n\varepsilon\,\|x\|^s,(n-6)\,|\,n^3-n^s\,|\,r\right), \\ N'\left(\varepsilon\,\|x\|^{ns},(n-6)\,|\,n^3-n^{ns}\,|\,r\right), \\ N'\left((n+1)\varepsilon\,\|x\|^{ns},(n-6)\,|\,n^3-n^{ns}\,|\,r\right) \end{cases} \quad \text{for all } x\in X \text{ and all } r>0.$$

Fuzzy Stability Results: Fixed Point Method: In this section, the authors presented the generalized Ulam - Hyers stability of the functional equation (1.6) in Fuzzy normed space using fixed point method.
Now we will recall the fundamental results in fixed point theory.

Theorem 5.1: [23] (The alternative of fixed point) Suppose that for a complete generalized metric space (*X, d*) and a strictly contractive mapping $T : X \to Y$ with Lipschitz constant L. Then, for each given element $x \in X$ either

$(B1)$ $d(T^n x, T^{n+1}x) = \infty \quad \forall \ n \geq 0,$ or

$(B2)$ there exists a natural number n_0 such that:

- $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
- The sequence $(T^n x)$ is convergent to a fixed point y^* of T
- y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;
- $d(y^*, y) \leq \frac{1}{1-L}\, d(y, Ty)$ for all $y \in Y$

In order to prove the stability result we define the following: δ_1 is a constant such that

$$\delta_i = \begin{cases} n & if \quad i=0, \\ \frac{1}{n} & if \quad i=1 \end{cases}$$

and Ω is the set such that $\Omega = \{g \mid g : X \to Y, g(0)=0\}$.

Theorem 5.2: *Let $f: X \to Y$ be a mapping for which there exist a function $\alpha : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x_1, \delta_i^k x_2, \cdots, \delta_i^k x_n\right), \delta_i^{3k} r\right) = 1, \tag{5.1}$$

for all $x_1, x_2, \cdots, x_n \in X, r > 0$ and satisfying the functional inequality

$$N\left(D\, f(x_1, x_2, \cdots, x_n), r\right) \geq N'\left(\alpha(x_1, x_2, \cdots, x_n), r\right), \tag{5.2}$$

for all $x_1, x_2, \cdots, x_n \in X, r > 0$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \frac{1}{n-6}\alpha\left(\frac{x}{n}, \frac{x}{n}, \cdots, \frac{x}{n}\right),$$

has the property

$$N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \forall\, x \in X, r > 0. \tag{5.3}$$

Then there exists unique cubic function $C: X \to Y$ satisfying the functional equation (1.6) and

$$N\left(f(x) - Q(x), r\right) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right), \forall\, x \in X, r > 0. \tag{5.4}$$

Proof: Let d be a general metric on Ω such that

$$d(g,h) = inf\begin{Bmatrix} K \in (0,\infty) \mid N\left(g(x) - h(x), r\right) \\ \geq N'\left(K\beta(x), r\right), x \in X, r > 0 \end{Bmatrix}.$$

It is easy to see that (Ω, d) is complete.

Define $T: \Omega \to \Omega$ by $Tg(x) = \frac{1}{\delta_i^3} g(\delta_i x),$ for all $x \in X$

For $g,h \in \Omega$, we have $d(g, h) \leq k$

$$\begin{aligned} &\Rightarrow\ N\left(g(x) - h(x), r\right) \geq N'\left(K\beta(x), r\right) \Rightarrow N\left(\frac{g(\delta_i x)}{\delta_i^3} - \frac{h(\delta_i x)}{\delta_i^3}, r\right) \geq N'\left(\frac{K}{\delta_i^3}\beta(\delta_i x), r\right) \\ &\Rightarrow\ N\left(Tg(x) - Th(x), r\right) \geq N'\left(KL\beta(x), r\right) \Rightarrow d\left(Tg(x), Th(x)\right) \leq KL \Rightarrow d\left(Tg, Th\right) \leq Ld(g,h) \end{aligned} \tag{5.5}$$

for all $g,h \in \Omega$ Therefore T is strictly contractive mapping on Ω with Lipschitz constant L Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, x, \cdots, x)$ in (5.2), we get

$$N\Big((n-6)f(nx) - n^3(n-6)f(x), r\Big) \geq N'\left(\alpha(x, x, \cdots, x), r\right). \tag{5.6}$$

for all $x \in X$, $r > 0$ Using (F3) in (5.6), we arrive

$$N\left(\frac{f(nx)}{n^3} - f(x), r\right) \geq N'\left(\frac{1}{n^3(n-6)}\alpha(x, x, \cdots, x), r\right) \tag{5.7}$$

for all $x \in X$, $r > 0$ with the help of (5.3) when $i = 0$, it follows from (5.7), we get

$$\begin{aligned}&\Rightarrow N\left(\frac{f(nx)}{n^3} - f(x), r\right) \geq N'\left(L\beta(x), r\right)\\&\Rightarrow d(Tf, f) \leq L = L^1 = L^{1-i}.\end{aligned} \tag{5.8}$$

Replacing x by $\frac{x}{n}$ in (5.6), we obtain

$$N\left(f(x) - n^3 f\left(\frac{x}{n}\right), r\right) \geq N'\left(\frac{1}{(n-6)}\alpha\left(\frac{x}{n}, \frac{x}{n}, \cdots, \frac{x}{n}\right), r\right) \tag{5.9}$$

for all $x \in X$, $r > 0$ with the help of (5.3) when $i = 0$, it follows from (5.9) we get

$$\begin{aligned}&\Rightarrow N\left(f(x) - n^3 f\left(\frac{x}{n}\right), r\right) \geq N'\left(\beta(x), r\right)\\&\Rightarrow d(f, Tf) \leq 1 = L^0 = L^{1-i}.\end{aligned} \tag{5.10}$$

Then from (5.8) and (5.10) we can conclude,

$$d(f, Tf) \leq L^{1-i} < \infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point C of T in Ω such that

$$C(x) = N - \lim_{k \to \infty} \frac{f(n^k x)}{n^{3k}}, \qquad \forall x \in X, r > 0. \tag{5.11}$$

Replacing $(x_1, x_2, \cdots, x_n)$ by $(\delta_i x_1, \delta_i x_2, \cdots, \delta_i x_n)$ in (5.2), we arrive

$$\begin{aligned}&N\left(\frac{1}{\delta_i^{3k}} Df(\delta_i x_1, \delta_i x_2, \cdots, \delta_i x_n), r\right)\\&\qquad \geq N'\left(\alpha(\delta_i x_1, \delta_i x_2, \cdots, \delta_i x_n), \delta_i^{3k} r\right)\end{aligned} \tag{5.12}$$

for all $r > 0$ and all $x_1, x_2, \cdots, x_n \in X$.

By proceeding the same procedure as in the Theorem 4.1, we can prove the function, $C : X \to Y$ satisfies the functional equation (1.6).

By fixed point alternative, since C is unique fixed point of T in the set

$$\begin{aligned}&\Delta = \left\{f \in \Omega \mid d(f, C) < \infty\right\},\\&\qquad \text{such that } N\left(f(x) - C(x), r\right) \geq N'\left(K\beta(x), r\right)\end{aligned} \tag{5.13}$$

for all $x \in X$, $r > 0$ and $K > 0$ Again using the fixed point alternative, we obtain $d(f, C) \leq \frac{1}{1-L} d(f, Tf)$

$$\Rightarrow d(f, C) \leq \frac{L^{1-i}}{1-L} \Rightarrow N\left(f(x) - C(x), r\right) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right), \tag{5.14}$$

for all $x \in X$ and $r > 0$. This completes the proof of the theorem.

From Theorem 5.2, we obtain the following corollary concerning the stability for the functional equation (1.6).

Corollary 5.3: *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\ldots,x_n),r\right) \geq \begin{cases} N'\left(\varepsilon,r\right), & \\ N'\left(\varepsilon\sum_{i=1}^{n}\|x_i\|^s,r\right), & s\neq 3; \\ N'\left(\varepsilon\prod_{i=1}^{n}\|x_i\|^s,r\right), & s\neq \frac{3}{n}; \\ N'\left(\varepsilon\left(\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right),r\right), & s\neq \frac{3}{n}; \end{cases} \tag{5.15}$$

for all $x_1,x_2,\cdots,x_n \in X$ and $r >$, where *ε, s* are constants with $\varepsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N\left(f(x)-C(x),r\right) \geq \begin{cases} N'\left(\varepsilon,(n-6)\,|\,n^3-1\,|\,r\right), \\ N'\left(n\varepsilon\,\|x\|^s,(n-6)\,|\,n^3-n^s\,|\,r\right), \\ N'\left(\varepsilon\,\|x\|^s,(n-6)\,|\,n^3-n^{ns}\,|\,r\right), \\ N'\left(\varepsilon(n+1)\,\|x\|^{ns},(n-6)\,|\,n^3-n^{ns}\,|\,r\right), \end{cases} \tag{5.16}$$

for all $x \in X$ and all $r > 0$.

Proof: Setting

$$\alpha(x_1,x_2,\cdots,x_n) = \begin{cases} \varepsilon, \\ \varepsilon\sum_{i=1}^{n}\|x_i\|^s, \\ \varepsilon\prod_{i=1}^{n}\|x_i\|^s, \\ \varepsilon\left(\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right). \end{cases}$$

for all $x_1,x_2,\cdots,x_n \in X$. Then,

$$N'\left(\alpha(\delta_i^k x_1, \delta_i^k x_2, \cdots, \delta_i^k x_n), \delta_i^{3k} r\right)$$
$$= \begin{cases} N'\left(\varepsilon, \delta_i^{3k} r\right) \\ N'\left(\varepsilon \sum_{i=1}^{n} \| x_i \|^s, \delta_i^{(3-s)k} r\right) \\ N'\left(\varepsilon \prod_{i=1}^{n} \| x_i \|^s, \delta_i^{(3-ns)k} r\right) \\ N'\left(\varepsilon \left(\prod_{i=1}^{n} \| x_i \|^s + \sum_{i=1}^{n} \| x_i \|^{ns}\right), \delta_i^{(3-ns)k} r\right) \end{cases}$$
$$= \begin{cases} \to 1 \; as \;\; k \to \infty, \\ \to 1 \; as \;\; k \to \infty, \\ \to 1 \; as \;\; k \to \infty, \\ \to 1 \; as \;\; k \to \infty. \end{cases}$$

Thus, (5.1) is holds. But we have

$\beta(x) = \frac{1}{n-6}\alpha\left(\frac{x}{n}, \frac{x}{n}, \cdots, \frac{x}{n}\right)$ has the property

$$N'\left(L \frac{1}{\delta_i^3} \beta(\delta_i x), r\right) \geq N'\left(\beta(x), r\right) \quad \forall \; x \in X, r > 0.$$

Hence

$$N'\left(\beta(x), r\right) = N'\left(\alpha\left(\frac{x}{n}, \frac{x}{n}, \cdots, \frac{x}{n}\right), (n-6) r\right)$$
$$= \begin{cases} N'\left(\varepsilon, (n-6) r\right), \\ N'\left(\frac{n\varepsilon}{n^s} \| x \|^s, (n-6) r\right), \\ N'\left(\frac{\varepsilon}{n^{ns}} \| x \|^s, (n-6) r\right), \\ N'\left(\frac{(n+1)\varepsilon}{n^{ns}} \| x \|^{ns}, (n-6) r\right). \end{cases}$$

Now,

$$N'\left(\frac{1}{\delta_i^3}\beta(\delta_i x), r\right) = \begin{cases} N'\left(\frac{\varepsilon}{\delta_i^3}, (n-6)r\right), \\ N'\left(\frac{\varepsilon}{\delta_i^3}\left(\frac{n}{n^s}\right)\|\delta_i x\|^s, (n-6)r\right), \\ N'\left(\frac{\varepsilon}{\delta_i^3}\left(\frac{1}{n^{ns}}\right)\|\delta_i x\|^s, (n-6)r\right), \\ N'\left(\frac{\varepsilon}{\delta_i^3}\left(\frac{n+1}{n^{ns}}\right)\|\delta_i x\|^{ns}, (n-6)r\right) \end{cases}$$

$$= \begin{cases} N'\left(\delta_i^{-3}\beta(x), r\right), \\ N'\left(\delta_i^{s-3}\beta(x), r\right), \\ N'\left(\delta_i^{ns-3}\beta(x), r\right), \\ N'\left(\delta_i^{ns-3}\beta(x), r\right). \end{cases}$$

Now from (5.4), we prove the following cases for conditions (i) and (I).

Case: 1 $L = n^{-1}$ if $i = 0$

$$N\left(f(x) - C(x), r\right) \ge N'\left(\frac{n^{-3}}{1-n^{-3}}\beta(x), r\right)$$
$$= N'\left(\frac{\varepsilon}{(n-6)(n^3-1)}\|x\|^s, r\right) = N'\left(\varepsilon\|x\|^s, (n-6)(n^3-1)r\right).$$

Case: 2 $L = n^3$ if $i = 0$

$$N\left(f(x) - C(x), r\right) \ge N'\left(\frac{1}{1-n^3}\beta(x), r\right)$$
$$= N'\left(\frac{\varepsilon}{(n-6)(1-n^3)}\|x\|^s, r\right) = N'\left(\varepsilon\|x\|^s, (n-6)(1-n^3)r\right).$$

Case: 3 $L = n^{s-3}$ for $s > 3$ if $i = 0$

$$N\left(f(x) - C(x), r\right) \ge N'\left(\frac{n^{s-3}}{1-n^{s-3}}\beta(x), r\right)$$
$$= N'\left(n\varepsilon\|x\|^s, (n-6)(n^3-n^s)r\right).$$

Case: 4 $L = n^{3-s}$ for $s > 3$ if $i = 0$

$$N\left(f(x) - C(x), r\right) \ge N'\left(\frac{1}{1-n^{3-s}}\beta(x), r\right)$$
$$= N'\left(n\varepsilon\|x\|^s, (n-6)(n^s-n^3)r\right).$$

Case: 5 $L = n^{ns-3}$ for $s > \frac{3}{n}$ if $i = 0$

$$N\left(f(x)-C(x),r\right) \geq N'\left(\frac{n^{ns-3}}{1-n^{ns-3}}\beta(x),r\right)$$
$$= N'\left(\varepsilon \|x\|^{s},(n-6)(n^{3}-n^{ns})r\right).$$

Case: 6 $L = n^{3-ns}$ for $s < \frac{3}{n}$ if $i = 0$

$$N\left(f(x)-C(x),r\right) \geq N'\left(\frac{1}{1-n^{3-ns}}\beta(x),r\right)$$
$$= N'\left(\varepsilon \|x\|^{s},(n-6)(n^{ns}-n^{3})r\right).$$

Hence the proof is complete.

Preliminaries of Random Normed Spaces: In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces as in [9, 38, 39].

Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F: R \cup \{-\infty,\infty\} \to [0,1]$, such that F is leftcontinuous and nondecreasing on $R, F(0)=0$ and $F(+\infty)=1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^{-}F(+\infty)=1$, where $l^{-}f(x)$ denotes the left limit of the function f at the point x that is, $l^{-}f(x) = \lim_{t\to x^{-}} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$ The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & if\ t \leq 0, \\ 1, & if\ t > 0. \end{cases} \tag{6.1}$$

Definition 6.1: *[38] A mapping $T:[0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm (briefly, a continuous t–norm) if T satisfies the following conditions:*

- T is commutative and associative;
- T is continuous;
- $T(a, 1) = a$ for all $a \in [0, 1]$;
- $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$

Typical examples of continuous $T_P(a,b) = ab, T_M(a,b) = min(a,b)$ norms are $T_L(a,b) = max(a+b-1,0)$ and $T_L(a,b) = max(a+b-1,0)$ (the Lukasiewicz n-norm). Recall (see [15, 16]) that if T is a t-norm and x_n is a given sequence of numbers in [0,1] then $T_{i=1}^{n} x_{n+i}$ is defined recurrently by

$T_{i=1}^{1} x_i = x_1$ *and* $T_{i=1}^{n} x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right)$ *for* $n \geq 2$. $T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$. It is known [16] that, for the Lukasiewicz t-norm, the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty \tag{6.2}$$

Definition 6.2: *[39] A random normed space (briefly, RN-space) is a triple* (X, μ, T)*, where* X *is a vector space,* T *is a continuous t-norm and* μ *is a mapping from* X *into* D^+ *satisfying the following conditions:*

- $\mu_x(t) = \varepsilon_0(t)$ for all $r > 0$ if and only if $x = 0$;
- $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
- $\mu_{x+y}(t+s) \geq T\left(\mu_x(t), \mu_y(s)\right)$ for all $x, y \in X$ and $t,s \geq 0$.

Example 6.3: *Every normed spaces* $(X, \|\cdot\|)$ *defines a random normed space* (X, μ, T_M)*, where*

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 6.4: *Let* (X, μ, T) *be a RN-space.*

- A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n \geq N$.
- A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq m \geq N$.
- A RN-space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X.

Theorem 6.5: *If* (X, μ, T) *is a RN-space and* $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$*, then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ *almost everywhere.*

Random Stability Results: Direct Method: In this section, the generalized Ulam - Hyers stability of the Cubic functional equation (1.6) in RN-space is provided. Throughout this section, let us consider X be a linear space and (X, μ, T) is a complete RN-space. The proof of the following Theorem and Corollary is similar to that of results of the Section 4 and 5. Hence the details of the proof are omitted.

Theorem 7.1: *Let* $j = \pm 1$*. Let* $f : X \to Y$ *be a mapping for which there exist a function* $\eta : X^n \to D^+$ *with the condition*

$$\begin{aligned} &\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{n^{(k+i)j}x_1, n^{(k+i)j}x_2, \cdots, n^{(k+i)j}x_n}\left(n^{3(k+i+1)j}\, t\right)\right) = 1 \\ &= \lim_{k\to\infty} \eta_{n^{kj}x_1, n^{kj}x_2, \cdots, n^{kj}x_n}\left(n^{3kj}\, t\right) \end{aligned} \tag{7.1}$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $t > 0$. such that the functional inequality with $f(0) = 0$ such that

$$\mu_{Df(x_1, x_2, \cdots, x_n)}(t) \geq \eta_{x_1, x_2, \cdots, x_n}(t) \tag{7.2}$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation (1.6) and

$$\mu_{C(x) - f(x)}(t) \geq T_{i=0}^{\infty}\left(\eta_{n^{(i+1)j}x, n^{(i+1)j}x, \cdots, n^{(i+1)j}x}\left(n^{3(i+1)j}\, t\right)\right) \tag{7.3}$$

for all $x \in X$ and all $t > 0$. The mapping $C(x)$ is defined by

$$\mu_{C(x)}(t)=\lim_{k\to\infty}\mu_{\frac{f(n^{kj}x)}{n^{3kj}}}(t) \tag{7.4}$$

for all $x \in X$ and all $t > 0$.

Proof: Assume $j = 1$. Setting $(x_1,x_2,\cdots,x_n)=(x,x,\cdots,x)$ in (7.1), we get

$$\mu_{(n-6)f(nx)-(n-6)n^3 f(x)}(t)\ge \eta_{x,x,\ldots,x}(t) \tag{7.5}$$

for all $x \in X$ and all $t > 0$. It follows from (7.5) and ($RN2$), we have

$$\mu_{\frac{f(nx)}{n^3}-f(x)}(t)\ge \eta_{x,x,\ldots,x}((n-6)n^3 t) \tag{7.6}$$

for all $x \in X$ and all $t > 0$. Replacing x by $n^k x$ in (7.6), we arrive

$$\mu_{\frac{f(n^{k+1}x)}{n^{3(k+1)}}-\frac{f(n^k x)}{n^{3k}}}(t)\ge \eta_{n^k x,n^k x,\ldots,n^k x}((n-6)n^{3(k+1)}t) \tag{7.7}$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to that of Theorem 4.1.

The following Corollary is an immediate consequence of Theorem 7.1, concerning the stability of (1.6).

Corollary 7.2: *Let ε and s be nonnegative real numbers. Let a cubic function $f : X \to Y$ satisfies the inequality*

$$\mu_{Df(x_1,x_2,\ldots,x_n)}(t)\ge\begin{cases}\eta_{\varepsilon}(t), & \\ \eta_{\varepsilon\sum_{i=1}^{n}\|x_i\|^s}(t), & s\ne 3;\\ \eta_{\varepsilon\prod_{i=1}^{n}\|x_i\|^s}(t), & s\ne \frac{3}{n};\\ \eta_{\varepsilon\left(\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right)}(t), & s\ne\frac{3}{n};\end{cases} \tag{7.8}$$

for all $x_1,x_2,\cdots,x_n\in X$ and all $t > 0$. Then there exists a unique cubic function $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t)\le\begin{cases}\eta_{\frac{\varepsilon}{(n-6)|n^3-1|}}(t),\\ \eta_{\frac{n\varepsilon\|x\|^s}{(n-6)|n^3-n^s|}}(t),\\ \eta_{\frac{\varepsilon\|x\|^{ns}}{(n-6)|n^3-n^{ns}|}}(t),\\ \eta_{\frac{\varepsilon\|x\|^{ns}}{(n-6)}\left(\frac{n+1}{|n^3-n^{ns}|}\right)}(t),\end{cases} \tag{7.9}$$

for all $x \in X$ and all $t > 0$.

Random Stability Results: Fixed Point Method: In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (1.6) in Random normed space using fixed point method.

Theorem 8.1: *Let* $f: X \to Y$ *be a mapping for which there exist a function* $\eta: X^n \to D^+$ *with the condition*

$$\lim_{k\to\infty} \eta_{\delta_i^k x_1, \delta_i^k x_2, \cdots, \delta_i^k x_n}\left(\delta_i^{3k} t\right) = 1, \ \forall\, x_1, x_2, \cdots, x_n \in X, t > 0 \tag{8.1}$$

and satisfying the functional inequality

$$\mu_{Df(x_1, x_2, \cdots, x_n)}(t) \ge \eta_{x_1, x_2, \cdots, x_n}(t), \ \forall\, x_1, x_2, \cdots, x_n \in X, t > 0. \tag{8.2}$$

If there exists $L = L(i)$ such that the function

$$x \to \beta(x,t) = \eta_{\frac{x}{n}, \frac{x}{n}, \ldots, \frac{x}{n}}((n-6)t),$$

has the property

$$\beta(x,t) \le L \frac{1}{\delta_i^3} \beta\left(\delta_i x, t\right), \ \forall\, x \in X, t > 0. \tag{8.3}$$

Then there exists a unique cubic function $C: X \to Y$ satisfying the functional equation (1.6) and

$$\mu_{C(x)-f(x)}\left(\frac{L^{1-i}}{1-L} t\right) \ge \beta(x,t), \ \forall\, x \in X, t > 0. \tag{8.4}$$

Proof: Let d be a general metric on Ω such that

$$d(g,h) = inf\begin{Bmatrix} K \in (0,\infty) \mid \mu_{g(x)-h(x)}(Kt) \\ \ge \beta(x,t), x \in X, t > 0 \end{Bmatrix}.$$

It is easy to see that (Ωd) is complete. Define $T: \Omega \to \Omega$ by $Tg(x) = \frac{1}{\delta_i^3} g(\delta_i x),$ for all $x \in X$ Now for $g,h \in \Omega$, we have $d(g,h) \le k$

$$\begin{aligned} d(g,h) \le K \Rightarrow\ & \mu_{g(x)-h(x)}(Kt) \ge \beta(x,t) \\ & \Rightarrow \mu_{Tg(x)-Th(x)}\left(\frac{Kt}{\delta_i^3}\right) \ge \beta(x,t) \\ & \Rightarrow d\left(Tg(x), Th(x)\right) \le KL \\ & \Rightarrow d\left(Tg, Th\right) \le L d(g,h) \end{aligned} \tag{8.5}$$

for all $g,h \in \Omega$ Therefore T is strictly contractive mapping on Ω with Lipschitz constant L The rest of the proof is similar to that of Theorem 5.2.

From Theorem 8.1, we obtain the following corollary concerning the stability for the functional equation (1.6).

Corollary 8.2: *Suppose that a function $f: X \to Y$ satisfies the inequality*

$$\mu_{Df(x_1,x_2,\dots,x_n)}(t) \geq \begin{cases} \eta_{\varepsilon}(t), & \\ \eta_{\varepsilon \sum_{i=1}^{n} \|x_i\|^s}(t), & s \neq 3; \\ \eta_{\varepsilon \prod_{i=1}^{n} \|x_i\|^s}(t), & s \neq \frac{3}{n}; \\ \eta_{\varepsilon \left(\prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns} \right)}(t), & s \neq \frac{3}{n}; \end{cases} \tag{8.6}$$

for all $x_1, x_2, \cdots, x_n \in X$ and $t > 0$, where ε, s are constants with $\varepsilon > 0$. Then there exists a unique cubic mapping $C: X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\varepsilon}{(n-6)|n^3-1|}}(t), \\ \eta_{\frac{n\varepsilon}{(n-6)|n^3-n^s|}\|x\|^s}(t), \\ \eta_{\frac{\varepsilon}{(n-6)|n^3-n^{ns}|}\|x\|^{ns}}(t), \\ \eta_{\frac{(n+1)\varepsilon}{(n-6)|n^3-n^{ns}|}\|x\|^{ns}}(t) \end{cases} \tag{8.7}$$

for all $x \in X$ and all $t > 0$.

Proof: Setting

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\varepsilon}{(n-6)|n^3-1|}}(t), \\ \eta_{\frac{n\varepsilon}{(n-6)|n^3-n^s|}\|x\|^s}(t), \\ \eta_{\frac{\varepsilon}{(n-6)|n^3-n^{ns}|}\|x\|^{ns}}(t), \\ \eta_{\frac{(n+1)\varepsilon}{(n-6)|n^3-n^{ns}|}\|x\|^{ns}}(t) \end{cases}$$

for all $x_1, x_2, \cdots, x_n \in X$ and all $t > 0$. The rest of the proof is similar to that of Corollary 5.3.

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