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Subalgebras of the Free Heyting Algebra on One Generator

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Abstract: In this paper we describe the subalgebras of the free Heyting algebra A_1 on one generator, generated by an arbitrary single element of A_1 and we give a general theorem which provides an explicit classification of all the subalgebras of A_1.

Key words: Free Heyting algebra · Subalgebra

INTRODUCTION

Heyting algebras are a generalization of Boolean algebras; the most typical example is the lattice of open sets of a topological space. It is well known that Heyting algebras are algebraic models of intuitionistic propositional logic, which is properly contained in classical propositional logic. Heyting algebras are special distributive lattices and they form a variety.

Definition: An algebra A = (A, $\vee$, $\wedge$, $\rightarrow$, 0, 1) is a *Heyting algebra* bounded distributive lattice with least element 0 and greatest element 1 and For all $x, y \in A$, $x \rightarrow y$ is the greatest element z of A such that $x \wedge z \leq y$, (wkere $\leq$ *is defined by*: $x \leq y$ *if and only if* $x \wedge y = x$). This element $x \rightarrow y$ is called the *pseudo-complement* of x with respect to.

In any Heyting algebra the pseudo-complement $\neg x$ of x is defined by $\neg x = x \rightarrow 0$. Note that $x \wedge \neg x = 0$ and that $\neg x$ is the greatest element having this property.

A complete Heyting algebra is a Heyting algebra that is a complete lattice (every subset has a supremum).

Let V be a variety. Recall that an algebra A $\in V$ is said to be a *free algebra* over V, if there exists a set $E \subseteq A$ such that E generates A and every mapping from E to an algebra B $\in V$ can be extended uniquely to a homomorphism from A to B. In this case E is said to be a set of free *generators*of A. If E is finite then A is said to be a *finitely generated free algebra.* We denote a *finitely generated free Heyting algebra* with α free generators (which is uniquelydetermined up to isomorphism) by A_α. Free Heyting algebras arise naturally as the Lindenbaumalgebras of intuitionistic propositional logic (IPC) with α propositional variables over the emptytheory.

In contrast with Boolean algebras, finitely generated free Heyting algebras are infinite as wasshown by Mckinsey and Tarski in the 1940s ([1]). For one generator, A_1 is well understood (The Rieger Nishimura ladder), but from two on, the structure remains mysterious, though manyproperties are known.

With the help of recursively described *Kripkemodels* H_α, Bellissima ([2]) gave a representationof A_α. Essentially the same construction is due to Grigolia ([3]) and Esakia.

The free Heyting algebra A_1 on one generators may be defined as the Lindenbaum algebra ofintuitionistic propositional logic *IPC* on a set $P = P_1$ of one propositional variable, this is theso-called `Rieger-Nishimura lattice', or `ladder', [4, 5], has an explicit description as a lattice; allelements which are not its least or greatest elements 0,1, lie in antichains of size 2, of which thereare countably many arranged in ω levels and the partial order relation between these is quite easilyand explicitly described.

Our aim in this paper is to provide an explicit classification of all the subalgebras of A_1.

Definition: Let A be an algebra, $X \subseteq$ A. The subalgebra generated by X is the intersection of allalgebras containing A written $\langle X \rangle$. (This can be constructed by closing up X under the operations.)

The Rieger-Nishimura Ladder, A_1, is shown in Figure 1.

Subalgebras of A_1: First I give the subalgebras of A_1 which are generated by only one element.

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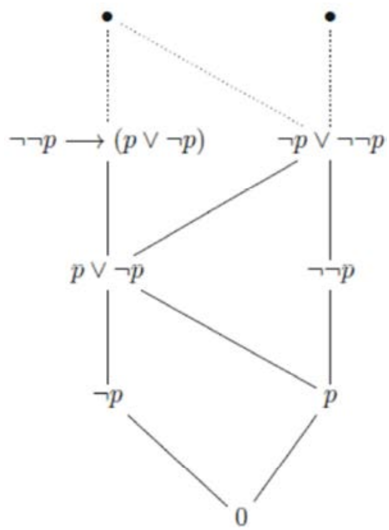

Fig. 1:

Example 2.1: S is the subalgebra generated by either 0 or 1.

Then S contains only {0, 1} see Figure 1.

Example 2.2: S is the subalgebra generated by *p*.

Then S equals A_1 see Figure 1.

Example 2.3: The set $\{0, \neg_p\ \neg\neg_p, \neg_p \vee \neg\neg_p, 1\}$ as in Figure 2 is equal to the subalgebra *S* generated by either $\neg_p$ *or* $\neg\neg_p$.

Indeed, it is closed under $\{\neg, \wedge, \vee\ \neg\}$, since $\neg\neg_p = \neg_p$, $\neg_p \wedge \neg\neg_p = 0$, $\neg_0 = 1$ and Table 1shows it is closed under $\neg$.

Theorem 2.4: If $x \in A_\alpha$ and *x is* above every atom, then $\neg_x = 0$.

Proof: Suppose that $\neg_x \neq 0$. Then $\neg_x \geq y$ for some atom *y*. But $x \geq y$, hence $x < \wedge \neg x \geq y$ whichis a contradiction.

Hence the negation of any element of $x \in A_1$ apart from 0, *p*, $\neg p$, $\neg\neg p$ is equal to 0.

Example 2.5: S is the subalgebra generated by such *x*.

Then S is the chain {0, *x*, 1} see Figure 4.

The previous examples gave a complete description of the subalgebras generated by arbitrary single element of A_1 and it follows from them that subalgebras of free algebras need not be free.

Now, in general I will use the following notation for the elements of A_1:

Fig. 2:

Table 1:

$\to$	0	$\neg p$	$\neg\neg p$	$\neg p \vee \neg\neg p$	1
0	1	1	1	1	1
$\neg p$	$\neg\neg p$	1	$\neg\neg p$	1	1
$\neg\neg p$	$\neg p$	$\neg p$	1	1	1
$\neg p \vee \neg\neg p$	0	$\neg p$	$\neg\neg p$	1	1
1	0	$\neg p$	$\neg\neg p$	$\neg p \vee \neg\neg p$	1

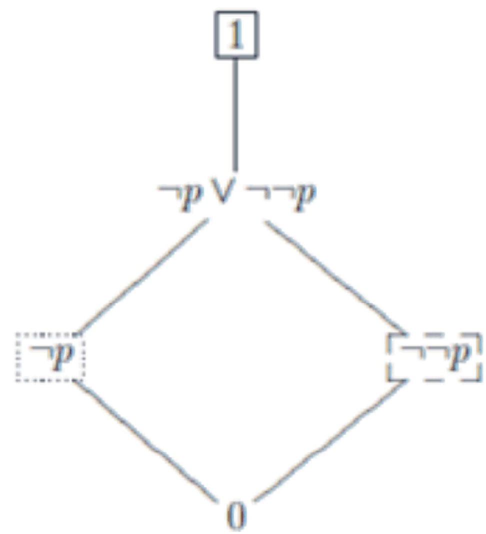

Fig. 3:

Fig. 4:

$0 = \perp, a_1 = \to_p, b_1 = p, a_{n+1} = a_n \to b_1,$
$b_{n+1} = a_n \vee b_n,$ and $1 = P \to P$

see Figure 5.

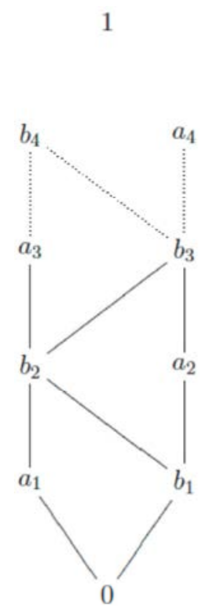

Fig. 5:

Note that, $\rightharpoondown a_1 = a_2$, $\rightharpoondown a_2 = a_1$ and $\rightharpoondown a_i = 0$ for all $I \geq 3$, $\rightharpoondown b_i = 0$ for all $i \geq 2$.

Lemma 2.6: The subalgebra S generated by each of the following sets:

1. $\{a_2, b_2\}$
2. $\{a_1, b_2\}$
3. $\{a_1, a_3\}$
4. $\{a_2, a_3\}$

is equal to A_1.

Proof: To prove this lemma it is enough to generate b_1 inside S.

1. $a_2 \wedge b_2 = b_1$
2. $\rightharpoondown a_1 = a_2$ and $a_2 \wedge b_2 = b_1$
3. $\rightharpoondown a_1 = a_2$, $a_1 \vee a_2 = b_3$, $a_3 \wedge b_3$ and $a_2 \wedge b_2 = b_1$
4. $a_2 \wedge a_3 = b_1$.n

Definition: We say that a subset X of A_1 is nearly A_1 if its complement is finite. A partially ordered set $(X, <)$ is quasi-linearly ordered if there is a linearly ordered set $(Y, <)$ and a homomorphism θ from X onto Y such that for each y, $\theta^{-1}(y)$ is either a singleton or a diamond.

Lemma 2.7: The subalgebra S generated by the set $\{b_n, b_{n+1}\}$ for $n \geq 2$ is nearly A_1.

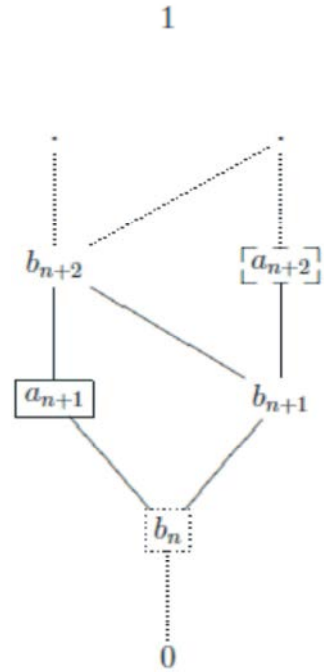

Fig. 6:

Proof: As b_n is above all the atoms, $\rightharpoondown b_n = 0$. Also $b_n \rightharpoondown b_{n+1} = 1$, $b_{n+1} \rightharpoondown b_n = a_{n+1}$, $a_{n+1} \vee b_{n+1} = b_{n+2}$ and $a_{n+1} \rightharpoondown b_{n+1} = a_{n+2}$ and so on, so we get all the points of A_1 above b_n as in Figure 6.

Corollary 2.8: The subalgebra S generated by the set $\{a_n, b_n\}$ fog $n \geq 3$ is nearly A_1.

Proof: This is an immediate consequence of the previous lemma, since $a_n \wedge b_n = b_{n-1}$.

Lemma 2.9: The subalgebra S generated by the set $\{a_{n+1}, b_n\}$ (or the set $\{\{a_{n+2}, b_n)$ for $n \geq 2$ is quasi-linearly ordered, with just one diamond.

Proof: As b_n is above all the atoms, $\rightharpoondown b_n = 0$. Also $a_{n+1} \rightharpoondown b_n = a_{n+2}$, $b_n \rightharpoondown a_{n+1} = 1$ and $a_{n+2} \rightharpoondown b_n = an_{+1}$, $b_n \rightharpoondown a_{n+2} = 1$ and $a_{n+1} \rightharpoondown a_{n+2} = a_{n+2}$, $a_{n+2} \rightharpoondown a_{n+1} = a_{n+1}$ and $a_{n+1} \vee a_{n+2} = b_{n+3}$. Hence we get 0, b_n, a_{n+1}, a_{n+2}, b_{n+3}, 1 which is a diamond starting at b_n, see Figure 6. To verify that it is subalgebra, note that it is clearly a lattice and it is also closed under $\rightharpoondown$ as shown in Table 2.

Corollary 2.10: The subalgebra S generated by the set $\{a_n, a_{n+1}\}$ for $n \geq 3$ is quasi-linearly ordered.

Proof: Since $a_n \wedge a_{n+1} = b_{n-1}$.

Table 2:

$\to$	0	b_n	a_{n+1}	a_{n+2}	b_{n+3}	1
0	1	1	1	1	1	1
b_n	0	1	1	1	1	1
a_{n+1}	0	a_{n+2}	1	a_{n+2}	1	1
a_{n+2}	0	a_{n+1}	a_{n+1}	1	1	1
b_{n+3}	0	b_n	a_{n+1}	a_{n+2}	1	1
1	0	b_n	a_{n+1}	a_{n+2}	b_{n+3}	1

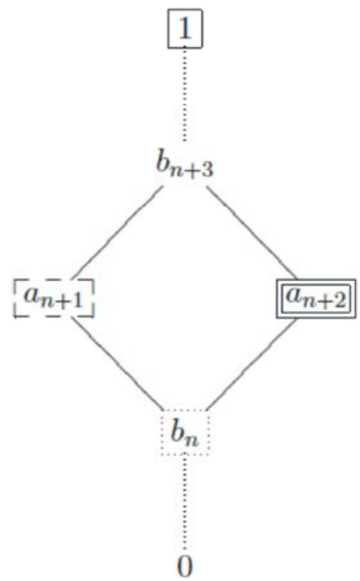

Fig. 7:

Theorem 2.11: The subalgebras of A_1 are all of one of the following forms:

- Nearly A_1(differs from A_1 on a finite set).
- Quasi-linearly ordered (linearly ordered and decorated by diamonds).

Proof: Let S be a subalgebra of A_1.

By Example 2.2 and Lemma 2.6 we may assume that S does not contain any of the sets $\{b_1\}$, $\{a_2, b_2\}$, $\{a_1, b_2\}$, $\{a_1, a_3\}$, or $\{a_2, a_3\}$ since then we obtain A_1 itself. If S contains $\{b_n, b_{n+1}\}$ for some $n \geq 2$, then we use Lemma 2.7 to see that S contains a subalgebrawhich is nearly A_1, so is nearly A_1 itself.

Now, suppose that S does not contain any of these subsets. Let x and y be any incomparable members of S. Then by inspection, $\{x, y\} = \{a_n, b_n\}$ of $\{a_n, a_{n+1}\{$ for some n Corollary 2.8 tells us about the first and Corollary 2.10 about the second.

Let D_1, D_2 be any two diamonds occurring in S. Then $D_1 = \{b_m, a_{m+1}, a_{m+2}, b_{m+3}\}$ and $D_2 = \{b_n, a_{n+1}, a_{n+2}, a_{n+2}, b_{n+3}\}$ for some m,n. Suppose that $m < n$. Then $n \neq m + 1$ or $m + 2$ as the former gives $\{b_m, b_{m+1}\} \subseteq$ S and the latter gives $\{b_n\ b_{n+1},\} \subseteq$ S, contrary to assumption. Therefore, $m + 3 \leq n$ and $D_1 \leq D_2$. Since all points of S not in diamonds are comparable with all members of S it follows that S is quasi-linearly ordered.

Now using Theorem 2.11 we can explicitly describe all subalgebras of A_1. If S is nearly A_1 thenthe least $x \in$ S such that for all $y \geq x$, $y \in$ S must equal b_n for some n (where this includes thepossibility of $n = 0$ where $b_n = 0$) and then since $a_n \wedge b_n = b_{n-1}$, for $n > 0$ we have $a_n \notin$ S. Using the proof of Theorem 2.11 we can then see that the $\{y \in$ S: $y \leq x\}$ is quasi-linearly ordered. Which quasi-linearly ordered sets can arise is restricted in a similar way to what follows.

To complete our description we must just characterize which S arise as quasi-linearly ordered.

Let X, Y and Z be three disjoint subsets of n: $n \geq 2$ satisfying:

- If $m < n$ in X then $n = m + 3$ or $n > m + 4$.
- If $n \in X$, $m \in Y$ then $n \neq m \pm 1$and $n = 3 \neq m \pm 1$.
- If $n \in X$, $m \in Z$ then $m \neq n \pm 1$ and $m \neq n \pm 2$.
- If $m < n$ in $n \neq m + 1$ then.
- If $m < n$ in Z then $n \neq m + 1$.
- If $m < n$ such that $m \in Y$ and $n \in Z$ then $n \neq m + 1$ and $n \neq m + 2$.
- If $m < n$ such that $m \in Z$ and $n \in Y$ then $m2$.

To see where these clauses come from, suppose that S is quasi-linearly ordered and let.

$X = \{n$: b_n id the least point of a diamond of S$\}$,
$Y = \{n \notin X\ U\ (X + 3)$: $b_n \in$ S$\}$ and
$Z = \{n \notin (X + 1) \cup (X + 2)$: $a_n \in$ S$\}$, where $X + i = \{n + I$: $n \in X\}$.

Thus the diamonds of S are $\{b_n, a_{n+1}, a_{n+2}, b_{n+3}\}$ for $n \in X$, the isolated b_ns of S are b_n for $n \in Y$ and the isolated a_ns in S are a_n for $n \in Z$. Then the seven clauses are true as follows.

Clause 1: Says that two diamonds do not overlap, though they can `touch', but also they cannot beexactly 1 apart because of Lemma 2.7.

Clause 2: Says that there are no two consecutive $b_n s$ (again because of Lemma 2.7).

Clause 3: Similarly deals with b_n related to $a_{n\pm1}$, $a_{n\pm2}$ (using Lemma 2.9).

Clause 4: Says that two isolated $b_n s$ cannot be consecutive (Lemma 2.7).

Clause 5: Says that two isolated $a_n s$ cannot be consecutive (Lemma 2.10).

Clause 6: Corresponds to Lemma 2.9.

Clause 7: Corresponds to the fact that $\rightarrow a_2 = a_1$.

The explanations just given show that if S is quasi-linearly ordered then *X*, *Y* and *Z* must filfil all these clauses. Conversely, suppose that all clauses are filfilled. Then S is quasi-linearly ordered and is subalgebra [6, 7]. It is obviously a sublattice, so we just need to show that it is closed under $\rightarrow$.

Now we have already seen that each diamond is closed under $\rightarrow$ Also if $\{b_n, a_{m+1}, a_{m+2}, b_{m+3}\} \cup \{b_n\} \subseteq S$ where $2 \leq m < n - 3$ then by the first clause $m < n - 4$ and $b_n \rightarrow b_m = b_m$, $b_n \rightarrow a_{m+1} = a_{m+1}$, $b_n \rightarrow a_{m+2} = a_{m+2}$ and $b_n \rightarrow b_{m+3} = b_{m+3}$, if $\{b_n\} \cup \{b_m, a_{m+1}, a_{m+2}, b_{m+3} \subseteq S\}$ where $2 \geq m < n$ then by the first clause $m < n - 1$ and $b_n \rightarrow b_m = b_m$, $a_{n+1} \rightarrow b_m = b_m$, $a_{n+2} \rightarrow b_m = b_m$ and $b_{n+3} \rightarrow b_m = b_m$.

If $\{b_n, a_{m+1}, a_{m+2}, b_{m+3}\} \cup \{a_n\} \subseteq S$ where $2 \leq m < n - 3$ then by the third clause $m < n - 5$ and $a_n \rightarrow b_m = b_m$, $a_n \rightarrow a_{m+1} = a_{m+1}$, $a_n \rightarrow a_{m+2} = a_{m+2}$ and $a_n \rightarrow b_{m+3} = b_{m+3}$, if $\{a_n\} \cup \{b_n, a_{n+1}, a_{n+2}, b_{n+3}\} \subseteq S$ where $2 < m < n$ then $b_n \rightarrow a_m = a_m$ $a_{n+1} \rightarrow a_m = a_m$ $a_{n+2} \rightarrow a_m = a_m$ and $b_{n+3} \rightarrow a_m = a_m$.

Finally, if $\{b_n\} \cup \{a_m\} \subseteq S$ where $2 \leq m < n$ then by clause 6, $m < n - 2$ and $a_n \rightarrow b_m = b_m$ and if $\{a_m\} \cup \{b_n\} \subseteq S$ where $2 < m < n$ then by clause 6, $m < n - 2$ and $b_n \rightarrow a_m = a_m$.

CONCLUSION

There are only countably many subalgebras which are nearly A_1 since there are only countably many finite sets which can be omitted and there are uncountably many subalgebras which are quasi-linearly ordered since if S is quasi-linearly ordered and infinite then we can omit any set of isolated points or diamonds in 2^{No} ways and obtain other quasi-linearly ordered sub algebras.

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