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Piecewise Quadratic Polynomial Solution for Boundary Value Problem of Fractional Order

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Abstract: Fractional differential equations have extensively used in physics, chemistry as well as engineering fields. Therefore, approximating the solution of differential equations of fractional order is necessary. Consequently, it is essential to approximate the solution of differential equations of fractional order. The piecewise quadratic polynomial function based method has been presented in this paper to find approximate solution for a class of boundary value problems of fractional order. Also we have obtained a consistency relation which can be applied in computing approximation to solve this group of boundary value problems. Finally four numerical examples are presented to describe the fractional usefulness of the suggested method.

Key words: Value problems of fractional order · Riemann–Liouville fractional derivative · Caputo fractional derivative · Piecewise quadratic polynomial

INTRODUCTION

In the past, it was believed that classical fractional calculus can provide a powerful tool that can be used to describe a large group of dynamic processes in various applied sciences, however it has been proved by more recent studies that fractional calculus can provide more accurate models compared with the classical fractional calculus. This is why fractional calculus has received agreat degree of interest in recent years. Fractional derivative and fractional integration have many applications in different complex systems such as physics, chemistry, fluid mechanics, viscoelasticity, signal processing, mathematical biology and bioengineering and various applications in many branches of science and engineering could be found [1-20] and bioengineering. Moreover they have various applications in different branches of science and engineering. Boundary value problems of fractional order are applied in accounting for various physical process of stochastic transport. Also they have application in investigating the liquid Filtration in a strongly porous (fractal) medium [21]. Moreover, boundary value problems with integral boundary conditions form a fascinating and important class of problems. The special cases of these problems include two, three, multipoint and nonlocal boundary value problems. Integral boundary conditions also play a part in population dynamic and cellular systems [6, 7]. Furthermore, they appear in the mathematical model created for a micro-electro-mechanical system (MEMS) instrument which basically has been developed to measure the viscosity of fluids that we face with during oil well exploration [10]. It has been argued that solution of fractional equations (FDES) is required in order to analysis and designs various systems. The methods in this category include Laplace and Fourier transforms, eigenvector expansion, method based on Daguerre integral formula, direct solution based on Grunewald Letnikov approximation, truncated Taylor series expansion and power series method [22-30]. For the purpose of solving (FDES) numerically several algorithms have been created. These include fractional Adams-Moulton methods, explicit Adams multistep methods, fractional difference methods, decomposition method, variational iteration method, least squares finite element solution, extrapolation method. Also, they include the Kansa method which is convenient, meshless method that has been applied in dealing with a variety of partial differential equation models [31, 32]. We can see in that there exist at least one solution of fractional two point boundary value problems.

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The present research paper is organized in following sections: in section 2 some definitions and theorem are presented which necessary for our work. In section 3, we establish the direction of the proposed method. In section 4, we have included some numerical results in order to illustrate the applications and usefulness of the suggested method [33].

Preliminaries and Notations: In this section, we give some definitions and properties of the fractional calculus.
Let $f(x)$ be a function defined on (a, b), then

Definition 1: [18] The Riemann–Liouville fractional derivative:

$$^{R}D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1} f(t)\,dt, \quad \alpha > 0, \quad m-1 < \alpha < m.$$

Definition 2: [18] The Riemann–Liouville fractional integral:

$$D_{a+}^{-\alpha} f(x) = D_a^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad \alpha > 0,$$

Where Γ is the gamma function?

Definition 3: [18, 26, 29] Right Riemann–Liouville fractional integral:

$$D_{b-}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad \alpha > 0.$$

Definition 4: [4] The Caputo fractional derivative:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s)\,ds, \quad \alpha > 0, \quad m-1 < \alpha < m.$$

The relation between the Riemann–Liouville operator and Caputo operator is given by:

$$D^{\alpha} f(x) = {}^{R}D^{\alpha}\left[f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}(x-a)^k f^{(k)}(a) \right], \quad \alpha > 0, \quad m-1 < \alpha < m.$$

Theorem 1: (Leibniz' formula) [21] (p. 75) Let $f(x)$ be continuous on $[0, t]$ and let $g(x)$ be analytical at a for all $a \in [0, t]$. Then, for $a > 0$ and $0 < a < t$:

$$D^{-\alpha} f(x) g(x) = \sum_{n=0}^{\infty} (-1)^n C_{\alpha}^{n} D^n f(x)\, {}^{R}D^{-\alpha-n} g(x),$$

$$^{R}D^{\alpha} f(x) g(x) = \sum_{n=0}^{\infty} C_{\alpha}^{n} D^n f(x)\, {}^{R}D^{\alpha-n} g(x).$$

where D^n is the ordinary differential operator and $C_{\alpha}^{n} = \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)}$.

Lemma 1: [12] If $f(x)$ is continuous and $\alpha, \beta, > 0$, then the following relationships hold:

(1) $^{R}D^{\alpha}\left(D^{-\beta} f(x)\right) = {}^{R}D^{\alpha-\beta} f(x)$,

(2) $D^{-\alpha}D^{-\beta} f(x) = D^{-\beta}D^{-\alpha} f(x) = D^{-\alpha-\beta} f(x)$,

(3) $D^{-\alpha} x^{m} = \dfrac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} x^{m+\alpha}$,

(4) $D^{-\alpha} \exp(ax) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x (x-t)^{\alpha-1} \exp(at)\,dt$

$= x^{\alpha} \exp(ax)\gamma(\alpha, ax)$,

where, $\gamma(a,ax) = \dfrac{1}{x^{\alpha}\Gamma(\alpha)}\displaystyle\int_0^x t^{\alpha-1}\exp(-at)\,dt$, is called incomplete gamma function.

Lemma 2: [26, 29] If $f(x)$ is continuous and $\alpha, < 1, \beta > 0$, then the right Riemann–Liouville fractional operators follow the following properties:

1) $D_{b-}^{-\alpha} D_{b-}^{-\beta} f(x) = D_{b-}^{-\beta} D_{b-}^{-\alpha} f(x) = D_{b-}^{-\alpha-\beta} f(x)$,

(2) $D_{b-}^{\alpha} D_{b-}^{-\alpha} f(x) = f(x)$,

(3) $D_{b-}^{-\alpha} (b-x)^{k} = \dfrac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}(b-x)^{k+\alpha}$, $k \in N$,

(4) $D_{1-}^{-\alpha} x^{k} = \dfrac{(1-x)^{\alpha}}{\Gamma(1+\alpha)}\left[1 + k!\displaystyle\sum_{m=1}^{k} \frac{(-1)^{m}\Gamma(1+\alpha)}{(4-m)!\Gamma(1+\alpha+m)}(1-x)^{m}\right]$, $k \in N$.

Theorem 2: [19] Let $f(x) \in C^m$ [0, 1] and $\alpha \in (m-1, m)$, $m \in N$ and $g(x) \in C^m$ [0, 1]. Then for $x \in [0, 1]$:

(1) $D^{\alpha}D^{-\alpha} g(x) = g(x)$,

(2) $D^{-\alpha} D^{\alpha} f(x) = f(x) - \displaystyle\sum_{k=0}^{m-1} \frac{x^k}{k!} f^{(k)}(0)$,

(3) $\displaystyle\lim_{x\to 0} D^{\alpha} f(x) = \lim_{x\to 0} D^{-\alpha} f(x) = 0$,

(4) If $\alpha_i \in (0,1], i = 1,2,\ldots,n$ with $\alpha = \displaystyle\sum_{i=1}^{n} \alpha_i$ are such that, for each k = 1,2,...,m–1 there exist $i_k < n$ with $\displaystyle\sum_{j=1}^{i_k} \alpha_j = k$, then the following composition rule holds: $D^{\alpha} = D^{\alpha_n} \ldots D^{\alpha_2} D^{\alpha_1} f(x)$.

Analysis of the Method: In order to describe our proposed method, we consider the numerical solution of the following fractional boundary value problem (FBVP):

$$D^{-\alpha} y''(x) + y\ p(x) = g(x), \quad 0 \le \alpha < 1, \quad x \in [a,b] \tag{3.1}$$

subject to boundary conditions:

$$y(a) = y(b) = 0, \tag{3.2}$$

in which the function $p(x)$ and $g(x)$ are continuous on the interval [a, b] and the operator D^{α} represents the Caputo fractional derivative. The analytical solution of (3.1–3.2) cannot be taken from for arbitrary choices $p(x)$ and $g(x)$. When α = 0, the problem (1.1) is shortened to the classical second order boundary value problem.

The main goal of this research work is to apply piecewise quadratic polynomial spline function to create a new numerical method for the FBVP (3.1–3.2). To do so, we firstly, convert the FBVPs in (3.1) into the following form [34]:

$$y''(x) + D^{\alpha}\, y\, p(x) = D^{\alpha} g(x),\ 0 \le \alpha < 1, \quad x \in [a, b]. \tag{3.3}$$

In the second step, we introduce a finite set of grid points x_i by driving the interval $[a, b]$ into n–equal parts:

$$x_i = a + i\,h,\ x_0 = a,\ x_n = b,\ h = \frac{b-a}{n},\ i = 0,1,2,\ldots,n. \tag{3.4}$$

Let $y(x)$ be the exact solution of (3.3) and Y_i be an approximation to $y_i = y(x_i)$ obtained by the piecewise quadratic polynomial passing through the points $[x_i, y_i]$ and $[x_{i+1}, y_{i+1}]$.

Consider that each piecewise quadratic polynomial segment Y_i has the form:

$$Y_i(x) = a_i(x - x_i)^2 + b_i(x - x_i) + c_i,\ i = 0,1,2,\ldots,n-1, \tag{3.5}$$

where a_i, b_i and c_i are constants to be determined of $3n$ equation following:

$$Y_{i-1}(x_i) = Y_i(x_i)\ ; i = 1,\ldots,n-1, \tag{3.6}$$

$$Y'_{i-1}(x_i) = Y'_i(x_i)\ ; i = 1,\ldots,n-1, \tag{3.7}$$

$$Y''_i + D^{\alpha} Y_i\ p(x) = D^{\alpha}\, g(x)\ ; i = 0,1,\ldots,n-1; x \in [x_i, x_{i+1}], \tag{3.8}$$

$$Y_0(a) = Y_{n-1}(b) = 0. \tag{3.9}$$

We obtained Simple form of (3.6) to (3.9) which is as follows:

$$\begin{cases} a_{i-1}h^2 + b_{i-1}h - c_{i-1} - c_i = 0; i = 1,\ldots,n-1, \\ 2a_{i-1}h + b_{i-1} - b_i = 0\ ; i = 1,\ldots,n-1, \\ a_i f_i(x) + b_i l_i(x) + c_i k_i(x) - r(x) = 0;\ i = 0,1,\ldots,n-1;\ x \in [x_i, x_{i+1}] \\ a_{n-1}h^2 + b_{n-1}h + c_{n-1} = 0, \\ c_0 = 0 \end{cases} \tag{3.10}$$

where

$$f_i(x) = 2 + \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-s)^{-\alpha}\Big((s-x_i)^2\, p'(s) + 2(s-x_i)\, p(s)\Big)ds, \tag{3.11}$$

$$l_i(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-s)^{-\alpha}\Big((s-x_i)^2\, p'(s) + p(s)\Big)ds, \tag{3.12}$$

$$k_i(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-s)^{-\alpha}\, p'(s)\, ds, \tag{3.13}$$

$$r(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-s)^{-\alpha}\, g'(s)\, ds\ . \tag{3.14}$$

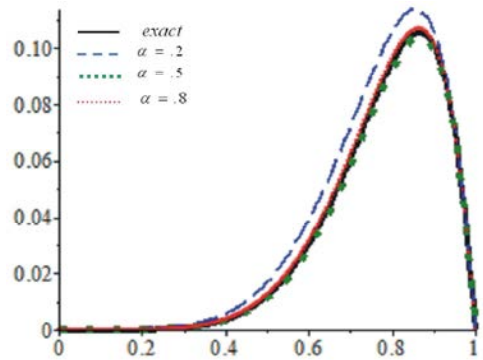

Fig. 1: Numerical solutions for various values of α *of Example 1*

Remark: According to the proposed method, it is clear that the equation system has a unique solution.

Illustrative Examples: We now consider some numerical examples [34] illustrating the solution using Piecewise quadratic polynomial methods. All calculations are implemented with MAPLE 16.

Example 1: Consider the boundary value problem:

$$D^{-\alpha} y''(x) + y(x) = \frac{720}{\Gamma(5+\alpha)} x^{4+\alpha} - \frac{40320}{\Gamma(7+\alpha)} x^{6+\alpha} + (1-x^2)x^6, \quad y(0) = y(1) = 0. \tag{4.1}$$

The analytical solution of (4.1) is

$$y(x) = x^6(1-x^2) \tag{4.2}$$

The numerical solutions for various values of α are represented in Fig.1.

Example 2: Consider the boundary value problem

$$D^{-\alpha} y''(x) + y(x) = ((-x^2 + (-3+2\alpha)x - \alpha^2)(x^\alpha \gamma(x,\alpha) + (x - x^2))e^x \quad y(0) = y(1) = 0. \tag{4.3}$$

The analytical solution of (4.3) is:

$$y(x) = x(1-x)e^x \tag{4.4}$$

The numerical solutions for various values of α are represented in Fig.2.

Example 3: Consider the boundary value problem:

$$D^{-\alpha} y''(x) + x\,y(x) = \frac{120}{\Gamma(4+\alpha)} x^{3+\alpha} - \frac{\Gamma(5+\alpha)}{\Gamma(3+2\alpha)} x^{2+2\alpha} + x^6 - x^{5+\alpha} \quad y(0) = y(1) = 0. \tag{4.5}$$

The analytical solution of (4.5) is:

$$y(x) = x^5\,(x - x^\alpha). \tag{4.6}$$

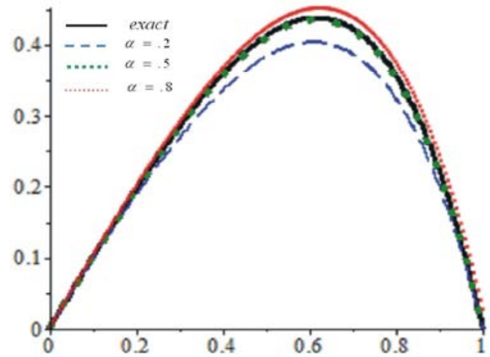

Fig. 2: Numerical solutions for various values of α *of Example 2*

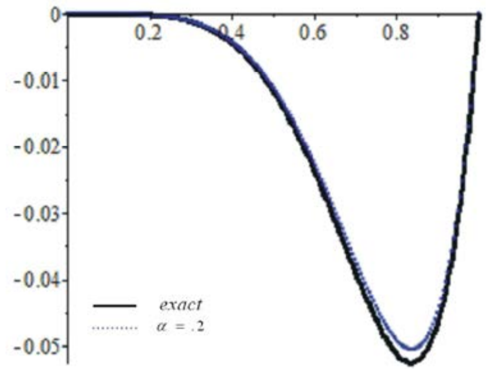

Fig. 3: Solution profiles for $\alpha = 0.2$ of *Example 3*

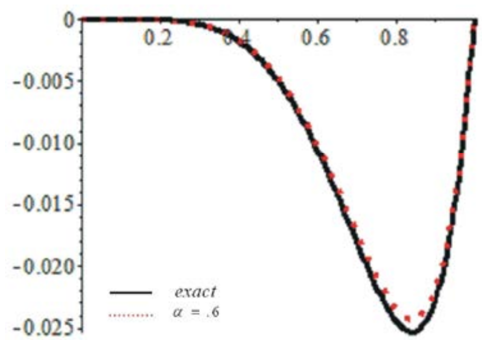

Fig. 4: Solution profiles for $\alpha = 0.6$ of *Example 3*

The numerical solutions for $\alpha = 0.2$ and $\alpha = 0.6$ are represented in Figs. 3 and 4.

Example 4: Consider the boundary value problem:

$$\begin{aligned} &D_{1-}^{-\alpha} y''(x) + y(x) = g(x), \\ &\quad y(0) = y(1) = 0. \end{aligned} \tag{4.7}$$

where

$$g(x) = x^6 - x^4 + \frac{(1-x)^{\alpha}}{\Gamma(1+\alpha)}\left[18 + 720\sum_{m=1}^{4}\frac{(-1)^m\,\Gamma(1+\alpha)}{(4-m)!\,\Gamma(1+\alpha+m)}(1-x)^m - 24\sum_{m=1}^{2}\frac{(-1)^m\,\Gamma(1+\alpha)}{(2-m)!\,\Gamma(1+\alpha+m)}(1-x)^m\right]$$

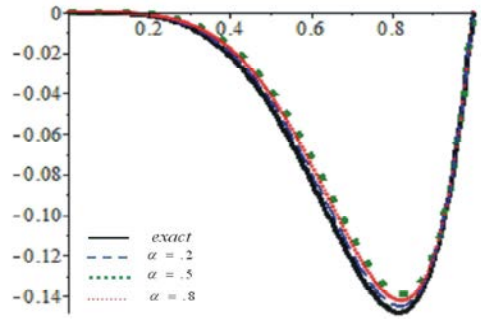

Fig. 5: The numerical solutions for various values of α of *Example 4*

The analytical solution of (4.7) is:

$$y(x) = x^4 (x^2 - 1). \tag{4.8}$$

The numerical solutions for various values of α are represented in Fig. 5.

CONCLUSION

We have presented a new method to solve fractional boundary value problem. Also we have presented convergent analysis for this method. The numerical results obtained in this paper indicate that the suggested method maintains a considerable degree of high accuracy which is promising in dealing with the solution of two point boundary value problem of fractional order.

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