Middle-East Journal of Scientific Research 21 (10): 1802-1808, 2014
ISSN 1990-9233

DOI: 10.5829/idosi.mejsr.2014.21.10.21744

Application of Optimal Homotopy Asymptotic Method to Magnetohydrodynamic Flow over a Nonlinear Stretching Sheet

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Abstract: In this paper, Magnetohydrodynamic (MHD) boundary layer is studied by using Optimal Homotopy Asymptotic Method (OHAM) and the solution of the governing Nonlinear Differential Equation is obtained by OHAM. This method is more efficient and flexible than the other methods such as the Homotopy Perturbation Method, Adomian Decomposition Method.

Key words: Optimal Homotopy Asymptotic Method · MHD flow · Nonlinear Stretching Sheet

INTRODUCTION

Plasma is a hot, ionized gas containing electrons and ions. The major use of MHD is in plasma Physics. The boundary layer flow of an incompressible viscous fluid over a continuously stretching sheet is often studied and encountered in many engineering techniques including aerodynamic extrusion of plastic sheets, hot rolling and glass -fiber production *et al*. [1-3].

The first work in this area was done by Sakiadis *et al*. [4, 5]. After that, various aspects of the stretching flow Problem were discussed by different investigators. Amongst these Chiam *et al*. [6] analysed the MHD flow of a viscous fluid bounded by stretching surface with power law velocity. He found the numerical solution of the boundary value problem by using the Runge-Kutta algorithm with Newton iteration. Now we investigate to analyze the MHD flow caused by a sheet with nonlinear stretching. The solution of the nonlinear problem is obtained by using Optimal Homotopy Asymptotic Method (OHAM).

Basic Idea of Oham: To explain the basic technique of the method, we consider the following Differential Equation:

$$L(F(z))+g(z)+N(F(z))=0,\ \mathcal{B}\left(\mathcal{F},\frac{d\mathcal{F}}{dz}\right)=0, \tag{2.1}$$

where L is a Linear Operator, z is independent variable, $F(z)$ is an unknown function, $g(z)$ is a known function, $N(F(z))$ is a nonlinear operator and B is a boundary operator.

According to technique, OHAM we construct a homotopy as, $s(\phi((z,p),p)):\mathcal{R}\times[0,1]\to\mathcal{R}$ which satisfies

$$(1-p)\left[\mathcal{L}(\phi(z,p))+g(z)\right]=\mathcal{H}(p)\left[\mathcal{L}(\phi(z,p))+g(z)+\mathcal{N}(\phi(z,p))\right]$$

$$\mathcal{B}\left(\phi(z,p),\frac{\partial\phi(z,p)}{\partial z}\right)=0, \tag{2.2}$$

where $z \in R$, $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$ and $\phi(z, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$, it holds that $\phi(z, 0) = F_0(z)$ and $\phi(z, 1) = F(z)$ respectively. Thus as p varies from 0 to 1, the solution $\phi(z, p)$ approaches from $F_0(z)$ to $F(z)$ where $F_0(z)$ is obtained from equation (2.2), for $p = 0$ and we have

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$$\mathcal{L}(\mathcal{F}_0(z)) + g(z) = 0, \mathcal{B}\left(\mathcal{F}_0, \frac{d\mathcal{F}_0}{dz}\right) = 0. \tag{2.3}$$

Next we choose auxiliary function $H(p)$ in the form

$$\mathcal{H}(p) = pC_1 + p^2 C_2 + \cdots \tag{2.4}$$

where C_1, C_2 . . . are constants to be determined latter. $H(p)$ can be expressed in many forms as investigated by Marinca *et al*. [7, 8]. To get an approximate solution one can expand $\phi(z, p, C_i)$ in Taylor's series about p in the following pattern:

$$\phi(z, p, C_i) = \mathcal{F}_0(z) + \sum_{k=1}^{\infty} \mathcal{F}_k(z, C_1, C_2 \cdots C_k) p^k \ , i = 1, 2 \cdots \tag{2.5}$$

Making use of equation (2.5) in equation (2.2) and comparing the coefficients of like powers of p, we get the following linear equations: ☻ zeroth order problem is given by equation (2.3) and the first and second order problems are given by equations (2.6) and (2.7) respectively.

$$\mathcal{L}(\mathcal{F}_1(z)) + g(z) = C_1 \mathcal{N}_0(\mathcal{F}_0(z)), \ \mathcal{B}\left(\mathcal{F}_1, \frac{d\mathcal{F}_1}{dz}\right) = 0, \tag{2.6}$$

$$\mathcal{L}(\mathcal{F}_2(z)) - \mathcal{L}(\mathcal{F}_1(z)) = C_2 \mathcal{N}_0(\mathcal{F}_0(z)) + C_1\left[\mathcal{L}(\mathcal{F}_1(z)) + \mathcal{N}_1(\mathcal{F}_0(z), \mathcal{F}_1(z))\right],$$
$$\mathcal{B}\left(\mathcal{F}_2, \frac{d\mathcal{F}_2}{dz}\right) = 0. \tag{2.7}$$

The general governing equations for $F_k(z)$ are given as

$$\mathcal{L}(\mathcal{F}_k(z)) - \mathcal{L}(\mathcal{F}_{k-1}(z)) = C_k \mathcal{N}_0(\mathcal{F}_0(z)) +$$
$$\sum_{i=1}^{k-1} C_i\left[\mathcal{L}(\mathcal{F}_{k-i}(z)) + \mathcal{N}_{k-i}(\mathcal{F}_0(z), \mathcal{F}_1(z), \cdots, \mathcal{F}_{k-1}(z))\right], k = 2, 3 \cdots, \mathcal{B}\left(\mathcal{F}_k, \frac{d\mathcal{F}_k}{dz}\right) = 0 \tag{2.8}$$

where $\mathcal{N}_m(\mathcal{F}_0(z), \mathcal{F}_1(z), \cdots, \mathcal{F}_{k-1}(z))$ is the coefficient of p^m in the expansion of $N(\phi(z, p))$ about the embedding parameter p

$$\mathcal{N}(\phi(z, p, C_i)) = \mathcal{N}_0(\mathcal{F}_0(z)) + \sum_{m=1}^{\infty} \mathcal{N}_m(\mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_m) p^m. \tag{2.9}$$

It has been observed that the convergence of the series (2.5) depends upon the auxiliary constants $C_1 C_2$, . . .
If it is convergent at $p = 1$ one has

$$\phi(z, C_i) = \mathcal{F}_0(z) + \sum_{k=1}^{\infty} \mathcal{F}_k(z, C_1, C_2, \cdots, C_k). \tag{2.10}$$

The result of the *mth* order approximations are given as

$$\tilde{\mathcal{F}}(z, C_1, C_2, \cdots, C_m) = \mathcal{F}_0(z) + \sum_{i=1}^{m} \mathcal{F}_i(z, C_1, C_2, \cdots, C_k). \tag{2.11}$$

Making use of equation (2.11) into equation (2.1) it results the following Residual

$$r(z, C_1, C_2 \cdots, C_m) = \mathcal{L}(\tilde{\mathcal{F}}(z, C_1, C_2, \cdots, C_m)) + g(z) + \mathcal{N}(\tilde{\mathcal{F}}(z, C_1, \cdots, C_m)). \tag{2.12}$$

If $r = 0$ then $\tilde{F}$ be the exact solution. Generally it does not happen especially in nonlinear problems. In order to get the optimal values of $C_i's$ where i=1, 2, 3, … We first construct the functional

$$J(C_1, C_2 \cdots, C_m) = \int_a^b r^2(z, C_1, C_2, \cdots, C_m)\, dz. \tag{2.13}$$

And then minimizing it we have

$$\frac{\partial J}{\partial C_1} = 0, \frac{\partial J}{\partial C_2} = 0, \cdots, \frac{\partial J}{\partial C_m} = 0, \tag{2.14}$$

where *a, b* are lying in the domain of the concerned problem. Making use of the Least Square Method we get the OHAM solution.

Mathematical Formulation of the Problem: For second grade Incompressible Homogeneous fluid, The Cauchy stress tensor has the form

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{3.1}$$

where

$$A_1 = \nabla V + (\nabla V)^{\mathrm{T}}, \tag{3.2}$$

$$A_2 = \frac{dA_1}{dt} + A_1(\nabla V) + (\nabla V)^T A_1, \tag{3.3}$$

where V denotes the velocity ∇V= gradient operator, $\frac{d}{dt}$ denotes material time differentiation and in equation (3.1). The Spherical stress $-pI$ is due to the constraint of incompressibility, μ is the viscosity α_1, α_2 are normal stress moduli and A_1, A_2 are the first two Rivlin-Ericksen Tensors. After prolonged discussion the sign of α_1 in equation (3.1) in critical review of Dunn and Rajagopal *et al*. [9]. It is concluded that equation (3.1) explains the basic model of the fluid completely. If the fluid modeled by equation (3.1) is in accord with thermodynamics in the sense that all motions of the fluid meet the Clausius-Duhem inequality and the supposition that the definite Helmholtz free energy of the fluid is a minimum when the fluid is locally at rest, then

$$\mu \geq 0, \alpha_1 \geq 0, \alpha_1 + \alpha_2 = 0. \tag{3.4}$$

We consider the second grade flow satisfying from equations (3.1) to (3.4) past a flat sheet coinciding with the plane $y = 0$. The flow is confined to $y > 0$. Two equal and opposite forces are applied along x-axis so that the wall is stretched keeping the origin fixed and a uniform magnetic field, B is applied along y-axis. The steady two dimensional boundary layer equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.5}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2(x)}{\rho}u + \lambda\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^3 u}{\partial y^3}\right], \tag{3.6}$$

where $v=\frac{\mu}{\rho}$ is the kinematic viscosity *u, v* are the velocity components in the *x* and *y* directions ρ is the fluid density. σ is the electrical conductivity of the fluid. *B* the uniform magnetic field along the *y*-axis and $\lambda=\frac{\alpha_1}{\rho}$, neglecting the induced magnetic field and using equations (3.1) to (3.4) we get from equations (3.5), (3.6) that

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.7}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=v\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2(x)}{\rho}u. \tag{3.8}$$

In equation (3.8) the external electric field and the polarization outcomes are negligible *et al*. [6]

$$B(x)=B_0 x^{\frac{n-1}{2}}. \tag{3.9}$$

The boundary conditions concerned the nonlinear stretching sheets are as

$$u(x,0)=cx^n,\, v(x,0)=0,\, u(x,y)\to 0 \ \text{as}\ y\to\infty. \tag{3.10}$$

Using the following substitutions

$$\eta=\sqrt{\frac{q(n+1)}{2v}}x^{\frac{n-1}{2}}y,\, u=qx^n f'(\eta),\, v=-\sqrt{\frac{qv(n+1)}{2}}x^{\frac{n-1}{2}}\left[f(\eta)+\frac{n-1}{n+1}\eta f'(\eta)\right]. \tag{3.11}$$

The resulting nonlinear differential equation is given as

$$f'''+ff''-\beta f'^2-Mf'=0, \tag{3.12}$$

with boundary conditions

$$f(0)=0,\, f'(0)=1,\, f'(\infty)=0, \tag{3.13}$$

where

$$\beta=\frac{2n}{n+1},\, M=\frac{2\sigma B_0^2}{\rho q(n+1)}. \tag{3.14}$$

Oham for the Proposed Problem: We find solutions of equations (3.12) and (3.13) Making use of the method OHAM described in section 2, we write equation (3.12) in the form

$$F'''(z)+F(z)F''(z)-\beta\left(F'(z)\right)^2-MF'(z)=0, \tag{4.1}$$

$$F(0)=0,\, F'(0)=1,\, F'(\infty)=0. \tag{4.2}$$

For simplicity and justification of the problem letting $\beta = 1$, $M = 1$ and for large value *b* such that equations (4.1) and (4.2) can be written as

$$F'''(z)+F(z)F''(z)-\left(F'(z)\right)^2-F'(z)=0, \tag{4.3}$$

$$F(0)=0,\, F'(0)=1,\, F'(b)=0. \tag{4.4}$$

Letting $b = 3$, the reason for $b = 3$ will be cleared latter on after displaying the graph.
The exact solution is given as

$$\mathcal{F}(z)=\frac{1-e^{-\sqrt{2}z}}{\sqrt{2}}. \tag{4.5}$$

Now applying the technique described in section 2 we have zeroth order problem which is given as

$$\mathcal{F}_0'''(z)=0,\mathcal{F}_0(0)=0,\mathcal{F}_0'(0)=1,\mathcal{F}_0'(3)=0, \tag{4.6}$$

Its solution is given as

$$\mathcal{F}_0(z)=\frac{1}{6}\left(6z-z^2\right). \tag{4.7}$$

First order problem is given as

$$\mathcal{F}_1'''(z,C_1)=-C_1\mathcal{F}_0'(z)-C_1\left(\mathcal{F}_0'(z)\right)^2+C_1\mathcal{F}_0(z)\mathcal{F}_0''(z)+\left(1+C_1\right)\mathcal{F}_0'''(z). \tag{4.8}$$

Its solution is given as

$$\mathcal{F}_1(z)=\frac{1}{2160}\left(2295z^2C_1-720z^3C_1+60z^4C_1-2z^5C_1\right). \tag{4.9}$$

Second order problem is given as

$$\mathcal{F}_2'''(z,C_1)=-C_1\mathcal{F}_1'(z)-2C_1\mathcal{F}_0'(z)\mathcal{F}_1'(z)+C_1\mathcal{F}_1(z)\mathcal{F}_0''(z)+C_1\mathcal{F}_0(z)\mathcal{F}_1''(z)+\left(1+C_1\right)\mathcal{F}_1'''(z). \tag{4.10}$$

Its solution is given as

$$\mathcal{F}_2(z)=\frac{1}{1088640}\left(\begin{array}{l}1156680z^2C_1-362880z^3C_1+30240z^4C_1-1008z^5C_1+3001050z^2C_1^2\\-362880z^3C_1^2-162540z^4C_1^2+29988z^5C_1^2-2016z^6C_1^2+24z^7C_1^2+z^8C_1^2\end{array}\right). \tag{4.11}$$

Third order problem is given as

$$\begin{aligned}\mathcal{F}_3'''(z,C_1)=&-C_1(\mathcal{F}_1'(z))^2-C_1\mathcal{F}_2'(z)-2C_1\mathcal{F}_0'(z)\mathcal{F}_2'(z)+C_1\mathcal{F}_2(z)\mathcal{F}_0''(z)+C_1\mathcal{F}_1(z)\mathcal{F}_1''(z)\\&+C_1\mathcal{F}_0(z)\mathcal{F}_2''(z)+\left(1+C_1\right)\mathcal{F}_2'''(z).\end{aligned} \tag{4.12}$$

Its solution is given as,

$$\mathcal{F}_3(z,C_1)=\frac{1}{1437004800}\left(\begin{array}{l}1526817600z^2C_1-479001600z^3C_1+39916800z^4C_1-1330560z^5C_1\\+7922772000z^2C_1^2-958003200z^3C_1^2-429105600z^4C_1^2+79168320z^5C_1^2\\-5322240z^6C_1^2+63360z^7C_1^2+2640z^8C_1^2+12955793103z^2C_1^3\\-479001600z^3C_1^3-874783800z^4C_1^3+53474850z^5C_1^3+11642400z^6C_1^3\\-2051280z^7C_1^3+96195z^8C_1^3-3960z^9C_1^3+352z^{10}C_1^3-12z^{11}C_1^3\end{array}\right) \tag{4.13}$$

We extend the case to 10[th] order problem that is $\mathcal{F}_{10}'''(z,C_1)$ and the solution to $F_{10}(z, C_1)$ using equations (4.7), (4.9), (4.11), (4.13),…., we get 10[th] order approximate solution by OHAM for $p = 1$ is

$$\tilde{\mathcal{F}}(z) = \mathcal{F}_0(z) + \mathcal{F}_1(z,C_1) + \cdots + \mathcal{F}_{10}(z,C_1). \tag{4.14}$$

Following the technique stated in section 2 and using the domain $a = 0$, $b = 3$ we have the residual

$$r = \tilde{\mathcal{F}}'''(z) + \tilde{\mathcal{F}}(z)\tilde{\mathcal{F}}''(z) - \left(\tilde{\mathcal{F}}'(z)\right)^2 - \tilde{\mathcal{F}}'(z). \tag{4.15}$$

It is minimized for $a = 0$ and $b = 3$ we obtained $C_1 = -0.224963746$, 10[th] order approximate solution is

$$\begin{aligned}\tilde{\mathcal{F}}(z) &= z - 0.695081z^2 - 0.307266z^3 - 0.0999529z^4 + 0.0271249z^5 - 0.006674z^6 + 0.00144874z^7 - \\ &0.000253849z^8 + 0.0000317145z^9 - 1.7842\times10^{-6}z^{10} - 3.15046\times10^{-7}z^{11} + 1.11882\times10^{-7}z^{12} - \\ &1.80811\times10^{-8}z^{13} + 1.66972\times10^{-9}z^{14} - 2.02575\times10^{-11}z^{15} - 2.30302\times10^{-11}z^{16} + 4.72825\times10^{-12}z^{17} - \\ &5.84603\times10^{-13}z^{18} + 5.20314\times10^{-14}z^{19} - 3.34913\times10^{-15}z^{20} + 1.30311\times10^{-16}z^{21} + 1.42628\times10^{-18}z^{22} - \\ &7.42834\times10^{-19}z^{23} + 6.98069\times10^{-20}z^{24} - 3.79885\times10^{-21}z^{25} - 9.29134\times10^{-23}z^{26} + 3.99944\times10^{-24}z^{27} - \\ &5.22371\times10^{-25}z^{28} + 2.52022\times10^{-26}z^{29} - 5.98356\times10^{-28}z^{30} + O(z^{31})\end{aligned} \tag{4.16}$$

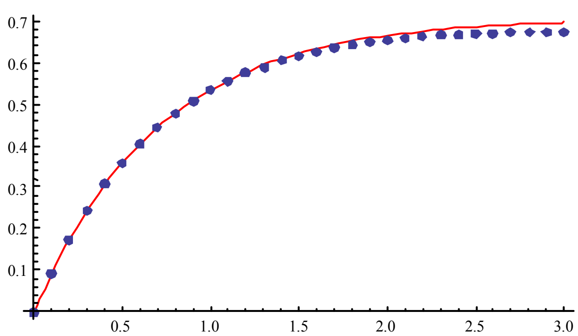

Fig. 1: Graph of exact solution and OHAM solution

Table 1: Displays values of exact solution (4.5) OHAM solution (4.16) and error.

z	Exact solution	OHAM solution	error
0	0	0	0
0.3	0.244481	0.244991	-5.0924 E-7
0.6	0.404434	0.405021	-5.87281 E-8
0.9	0.509083	0.508468	6.14847 E-8
1.2	0.577549	0.574594	2.95476 E-7
1.5	0.622344	0.616686	5.95808 E-7
1.8	0.65165	0.642463	9.18732 E-7
2.1	0.670824	0.658416	1.2407 E-5
2.4	0.683369	0.667751	1.5618 E-5
2.7	0.691576	0.672556	1.90198 E-6
3.0	0.696946	0.674004	2.29421 E-6

From the Table 1 we conclude that the exact and the OHAM solutions are converging up to the value $z = 3$ that is, there is a little difference between them. The two Graphs coincide up to $z = 3$ and after $z = 3$ the two Graphs are diverging that is why we have taken $b = 3$. From the Figure 1, we conclude that the exact and OHAM solutions are coincident, this means that the method OHAM is effective and reliable.

REFERENCES

1. Fisher, E.G., 1976. Extrusion of plastics, Wiley, New York.
2. Tadmor, Z. and I. Klein, 1970. Engineering principles of plasticating extrusion. Polymer Science and Engineering series. Reinhold, New York, Van Nostrand.
3. Altan, T. and S. Oh, 1979. Gegri, Metal Forming Fundamentals and Applications, American Society of Metals Park.
4. Sakiadis, B.C., 1961. Boundary-layer behavior on continuous solid surface, American Institute of Chemical Engineers Journal, 7: 26-28.
5. Sakiadis, B.C., 1961. Boundary layer behavior on continuous solid surface II: Boundary layer on a Continuous flat surface, American Institute of Chemical Engineers Journal, 7: 221-225.
6. Chaim, T.C., 1995. Hydromagnetic flow over a surface stretching with a power- law velocity. Int. J. Eng. Sci., 33: 429-435.
7. Marinca, V. and N. Herisanu, 2008. Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer, Int. Comm in Heat and Mass Transfer, 35: 710-715.

8. Marinca, V., N. Herisanu and L. Nemes, 2008. Optimal homotopy asymptotic method with application to thin film flow, Cent. Eur. J. Phys., 6(3): 648-653.
9. Dunn, J.E. and K.R. Rajagopal, 1995. Fluids of differential type: critical review and thermodynamic analysis, Int. J. Engng. Sci., 33: 689-729.
10. Rajagopal, K.R., 1995. On boundary conditions for fluids of differential type, In: A. Sequeira (Ed), Navier-Stokes Equations and Related Non linear Problems, Plenum Press, New York, pp: 273-278.
11. Papanastasiou, T.C., G.C. Georgiou and A.N. Alexandrou, 1994. Viscous Fluid Flow, CRC Press, pp: 413.
12. Ghori, Q.K., M. Ahmed and A.M. Siddiqui, 2007. Int. J. Nonlinear Sci. Num. Simul., 8(2): 179-184.
13. Vajravelu, K. and D. Rollins, 2004. Hydromagnetic flow of a second grade fluid over a stretching sheet, Applied Mathematics and Computation, 148: 783-791.
14. Hayat, T., Q. Hussain and T. Javed, 2007. Nonlinear Analysis. Real World Application, 10: 966-973.
15. Idrees, M., Sirajul Haq and S. Islam, 2010. Application of Optimal Homotopy Asymptotic Method to Fourth Order Boundary Value Problems, World Applied Sciences Journal, 9(2): 131-137.
16. Kechil, S.A. and I. Hashim, 2007. Series solution for unsteady boundary-layer flows due to impulsively stretching plate, Chin. Phys. Lett., 24: 139-142.
17. Kechil, S.A., I. Hashim and S.S. Jiet, 2007. Approximate analytical solution for a class of laminar boundary-layer equation, 24: 1981-1984.
18. Baker, G.A., 1975. Essentials of Pade Approximation, Academic press, London.
19. Boyd, J., 1997. Pade approximation algorithm for solving nonlinear ordinary differential equation value problems on an unbounded domain, Comput. Phys., 11: 299-303.
20. Pavlov, K.B., 1975. Magnetohydrodynamics flow of an incompressible viscous fluid caused by deformation of a surface, Magn. Gidrod, 4: 146-147.