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The Kumaraswamy-Generalized Lomax Distribution

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Abstract: The modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields. For the first time the Kum-GL distribution is introduced and studied. The new distribution can have a decreasing and upside-down bathtub failure rate function depending on the value of its parameters; it is including some special sub-model like Pareto Type I Distribution and its original form. Some structural properties of the proposed distribution are studied including explicit expressions for the moments. We provide the density function of the order statistics and obtain their moments. The method of maximum likelihood is used for estimating the model parameters and the observed information matrix is derived. A set of real data is provided to illustrate the theoretical results in the complete sampling case.

Key words: Hazard function · Kumaraswamy distribution · Moment · Maximum likelihood estimation · Lomax distribution.

INTRODUCTION

For life testing when the life times of items are continuous random variables, it is important to know the total number of individuals in the sample which is drawn from an assumed failure model, the total number of individuals may be unknown for many causes, either due to the omission in the records or perhaps because of physical conditions of the experiment and then the sample size should be estimated. The Lomax distribution (Pareto distribution of the second kind) has, in recent years, assumed apposition of importance in the field of life testing because of its uses to fit business failure data. The cumulative distribution function cdf of the parameters Lomax distribution is:

$$F(x;\alpha,\lambda) = 1-\left(1+\frac{x}{\lambda}\right)^{-\alpha} ,x>0 ,(\alpha,\lambda)>0 \qquad (1)$$

A random variable *x* is said to follow the Lomax distribution, if the probability density function pdf of *x* is as follows:

$$f(x;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} ,x>0 ,(\alpha,\lambda)>0 \qquad (2)$$

Where: λ is the scale parameter and α is the shape parameter.

In this context and based on the Kumaraswamy distribution [1]. We propose an extension of the Lomax distribution based on the family of Kumaraswamy generalized (denoted with the prefix "Kw-G" for short) distributions introduced by Cordeiro and de Castro [2]. Nadarajah *et al.* [3] studied some mathematical properties of this family. The Kumaraswamy (Kw) distribution is not very common among statisticians and has been little explored in the literature. Its cdf (for $0 < x < 1$) is $F(x) = 1-(1-x^a)^b$, where a > 0 and b > 0 are shape parameters and the density function has a simple form $f(x) = ab\,x^{a-1}(1-x^a)^{b-1}$, which can be unimodal, increasing, decreasing or constant, depending on the parameter values. It does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution been widely appreciated. However, in a very recent paper, Jones [1] explored the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and Kw distributions.

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In this note, we combine the works of Kumaraswamy [4] and Shawky *et al.* [5], to derive some mathematical properties of a new model, called the Kumaraswamy Generalized exponentiated Pareto (Kw-GEP) distribution, which stems from the following general construction: if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by:

$$F(x;a,b) = 1-(1-G(x)^{a})^{b} \tag{3}$$

Where: a > 0 and b > 0 are two additional shape parameters which govern skewness and tail weights. Because of its tractable distribution function (2), the Kw-G distribution can be used quite effectively even if the data are censored. Correspondingly, its density function is distributions has a very simple form

$$f(x;a,b) = ab\,g(x)G(x)^{a-1}(1-G(x)^{a})^{b-1} \tag{4}$$

The density family (3) has many of the same properties of the class of beta-G distributions [6], but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta-G family of distributions, special Kw-G distributions can be generated as follows: the Kw-normal distribution is obtained by taking G(x) in (2) to be the normal cumulative function. Analogously, the Kw-Weibull [7], Kw-generalized gamma [8], Kw-Birnbaum-Saunders [9] and Kw-Gumbel [2] distributions are obtained by taking G(x) to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, among several others. Hence, each new Kw-G distribution can be generated from a specified G distribution.

This paper is outlined as follows. In section 2, we define the KW-GL distribution and provide expansions for its cumulative and density functions. A range of mathematical properties of this distribution is considered in sections 3 and 4. These include quantile function, simulation, skewness and kurtosis. Maximum likelihood estimation is performed and the observed information matrix is determined in section 5. In section 6, we provide simulation study for the generated data. Finally, some conclusions are addressed.

The Kumaraswamy-Generalized Lomax Distribution: If $G(x;\underline{\theta})$ is the Lomax cumulative distribution with Paramete $\underline{\vartheta}=(\alpha,\lambda)$ then equation (1) yields the Kw-GL cumulative distribution for ($x \geq 0$)

$$F(x;\underline{\xi}) = 1-\left[1-\left(1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right)^{a}\right]^{b} \tag{5}$$

where $\underline{\xi}=(a,b,\alpha,\lambda),(a,b\,and\,\alpha)>0$ are non-negative shape Parameter and λ is the scale parameter. The corresponding pdf and Hazard Rate Function are:

$$f(x;\underline{\xi}) = ab\frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} \left(1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right)^{a-1}\left\{1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]^{a}\right\}^{b-1} \tag{6}$$

and

$$S(x;\underline{\xi}) = 1-F(x;\underline{\xi}) = \left[1-\left(1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right)^{a}\right]^{b},$$

$$H(x;\underline{\xi}) = \frac{f(x;\underline{\xi})}{S(x;\underline{\xi})} = \frac{ab\frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)}\left(1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right)^{a-1}}{1-\left(1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right)^{a}}$$

respectively

Special Distributions: The following well-known and new distributions are special sub-models of the Kum-GP distribution.

Exponentiated Lomax Distribution: If $b = 1$, the Kum-GL distribution reduces to

$$f(x;\underline{\xi}) = ab\frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)}\left(1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right)^{a-1}$$

Which is the exponentiated Lomax distribution (EL), for $a = b = 1$, we obtain the Lomax distribution.

Expansions for the Cumulative and Density Functions: Here, we give simple expansions for the Kw-GL cumulative distribution. By using the generalized binomial theorem (for 0 < a < 1)

$$(1+a)^{v} = \sum_{i=0}^{\infty}\binom{v}{i}a^{i}$$

In equation (5), we can write

$$F\left(x;\underline{\xi}\right)=1-\sum_{i=0}^{\infty}(-1)^i\binom{b}{i}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]^{ai}=1-\sum_{i=0}^{\infty}\eta_i\tau\left(x;\underline{\zeta}\right)$$

Where $\eta_i=(-1)^i\binom{b}{i}$ and $\tau\left(x;\underline{\zeta}\right)$ denotes the EL cumulative distribution with parameters $\underline{\zeta}=(ai,\alpha,\lambda)$, Now, using the power series (7) in the last term of (6), we obtain

$$f\left(x;\underline{\xi}\right)=\frac{b\alpha a\left(i+1\right)}{\lambda\left(i+1\right)}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}$$
$$\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]^{a\left(i+1\right)-1}$$

We can write

$$f\left(x;\underline{\xi}\right)=\sum_{i=0}^{\infty}k_i g(x;\underline{\vartheta}) \tag{8}$$

Where $k_i=\frac{b}{(i+1)}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}$ and $g\left(x;\underline{\vartheta}\right)$, denotes the Exponentiated Lomax Distribution with parameters $\underline{\vartheta}=\left(a\left(i+1\right),\alpha,\lambda\right)$. Thus, the Kw-GL density function can be expressed as an infinite linear combination of Lomax densities. Thus, some of its mathematical properties can be obtained directly from those properties of the Lomax distribution. For example, the ordinary, inverse and factorial moments, moment generating function (mgf) and characteristic function of the Kw-GL distribution follow immediately from those quantities of the Lomax distribution.

Quantile Function and Simulation: We present a method for simulating from the Kw-GL distribution (6). The quantile function corresponding to (5) is:

$$\mathrm{Q}\left(\mathrm{u}\right)=\mathrm{F}^{-1}\left(\mathrm{u}\right)=\left(\left[1-\left(1-\left(1-\mathrm{u}\right)^{\frac{1}{\mathrm{b}}}\right)^{\frac{1}{\mathrm{a}}}\right]^{\frac{-1}{\alpha}}-1\right)\lambda \tag{9}$$

Simulating the Kw-GL random variable is straight forward. Let U be a uniform variate on the unit interval (0,1). Thus, by means of the inverse transformation method, we consider the random variable X given by

$$X=\left(\left[1-\left(1-\left(1-u\right)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right]^{\frac{-1}{\alpha}}-1\right)\lambda$$

Which follows (6), i.e. $X\sim KW-GL\ (a,b,\alpha,\lambda)$

Skewness and Kurtosis: The shortcomings of the classical kurtosis measure are well-known. There are many heavy tailed distributions for which this measure is infinite. So, it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the non-existence of classical kurtosis for many of the Kw distributions. The Bowley's skewness [10] is based on quartiles:

$$\mathrm{S_k}=\frac{\mathrm{Q}_{3/4}-2\mathrm{Q}_{1/2}+\mathrm{Q}_{1/4}}{\mathrm{Q}_{3/4}-\mathrm{Q}_{1/4}}c$$

And the Moors' kurtosis [11] is based on octiles:

$$\mathrm{K_u}=\frac{\mathrm{Q}_{7/8}-\mathrm{Q}_{5/8}-\mathrm{Q}_{3/8}+\mathrm{Q}_{1/8}}{\mathrm{Q}_{6/8}-\mathrm{Q}_{2/8}}$$

Where $Q(.)$ represents the quantile function

Estimation and Information Matrix: In this section, we discuss maximum likelihood estimation and inference for the Kw-GL distribution. Let $x_1,x_2,\ldots,x_n$ be a random sample from $X\sim KW-GL\left(\underline{\xi}\right)$ where $\underline{\xi}=(a,b,\alpha,\lambda)$ be the vector of the model Parameters, the log-likelihood function for $\underline{\xi}$ reduces to

$$\begin{aligned}\ell\left(\underline{\xi}\right)&=n\log a+n\log b+n\log\alpha-n\log\\ \lambda\ &-(\alpha+1)\sum_{i=1}^{\infty}\log\left(1+\frac{x_i}{\lambda}\right)\\ &+(a-1)\sum_{i=1}^{\infty}\log\left(1-\left(1+\frac{x_i}{\lambda}\right)^{-\alpha}\right)+(b-1)\\ &\sum_{i=1}^{\infty}\log\left\{1-\left[1-\left(1+\frac{x_i}{\lambda}\right)^{-\alpha}\right]^{a}\right\}\end{aligned} \tag{10}$$

The score vector $U\left(\underline{\xi}\right)=\left(\partial\ell/\partial a,\partial\ell/\partial b,\partial\ell/\partial\alpha,\partial\ell/\partial\lambda\right)^T$, where the components corresponding to the parameters in $\underline{\xi}$ are given by differentiating (10). By setting $z_i=1-\left(1+\frac{x_i}{\lambda}\right)^{-\alpha}$

$$\frac{\partial\ell}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\log z_i-(b-1)\sum_{i=1}^{n}\frac{{z_i}^{a}\log z_i}{1-{z_i}^{a}}$$

$$\frac{\partial\ell}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log\left(1-{z_i}^{a}\right)$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \log\left(1+\frac{x_i}{\lambda}\right) + (a-1)\sum_{i=1}^{n} \frac{\left(1+\frac{x_i}{\lambda}\right)^{-\alpha} \log\left(1+\frac{x_i}{\lambda}\right)}{z_i}$$

$$- a(b-1)\sum_{i=1}^{n} \frac{{z_i}^{a-1}\left(1+\frac{x_i}{\lambda}\right)^{-\alpha} \log\left(1+\frac{x_i}{\lambda}\right)}{1-{z_i}^{a}}$$

and

$$\frac{\partial \ell}{\partial \lambda} = -\frac{n}{\lambda} + \frac{\alpha+1}{\lambda^2}\sum_{i=1}^{n} \frac{x_i}{\left(1+\frac{x_i}{\lambda}\right)} -$$

$$\frac{\alpha(a-1)}{\lambda^2}\sum_{i=1}^{n} \frac{x_i\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)}}{z_i} +$$

$$\frac{\alpha a(b-1)}{\lambda^2}\sum_{i=1}^{n} \frac{{z_i}^{a-1} x_i\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)}}{1-{z_i}^{a}}$$

The maximum likelihood estimates (MLEs) of the parameters are the solutions of the nonlinear equations $\nabla l = 0$, which are solved iteratively. The observed information matrix given

$$J_n\left(\underline{\xi}\right) = n\begin{pmatrix} J_{aa} & J_{ab} & J_{a\alpha} & J_{a\lambda} \\ J_{ba} & J_{bb} & J_{b\alpha} & J_{b\lambda} \\ J_{\alpha a} & J_{\alpha b} & J_{\alpha\alpha} & J_{\alpha\lambda} \\ J_{\lambda a} & J_{\lambda b} & J_{\lambda\alpha} & J_{\lambda\lambda} \end{pmatrix}$$

Whose elements are:

$$J_{aa} = -\frac{n}{a^2} - (b-1)\sum_{i=1}^{n} \frac{z_i^a \log^2 z_i}{\left(1-{z_i}^{a}\right)^2}$$

$$J_{ab} = -\sum_{i=1}^{n} \frac{z_i^a \log z_i}{1-{z_i}^{a}}$$

$$J_{a\alpha} = \sum_{i=1}^{n} \frac{\left(1+\frac{x_i}{\lambda}\right)^{-\alpha} \log\left(1+\frac{x_i}{\lambda}\right)}{z_i} - (b-1)$$

$$\sum_{i=1}^{n} \frac{z_i^{a-1}\left(1+\frac{x_i}{\lambda}\right)^{-\alpha} \log\left(1+\frac{x_i}{\lambda}\right)\left(a\log z_i - z_i^a + 1\right)}{\left(1-{z_i}^{a}\right)^2}$$

$$J_{a\lambda} = -\frac{\alpha}{\lambda^2}\sum_{i=1}^{n} \frac{x_i\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)}}{z_i} - \frac{\alpha(b-1)}{\lambda^2}$$

$$\sum_{i=1}^{n} \frac{z_i^{a-1} x_i\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)}\left[1+a\log z_i - z_i^a\right]}{\left(1-{z_i}^{a}\right)^2}$$

$$J_{bb} = -\frac{n}{b^2}\, J_{b\alpha} = -a\sum_{i=1}^{n} \frac{z_i^{a-1}\left(1+\frac{x_i}{\lambda}\right)^{-\alpha} \log\left(1+\frac{x_i}{\lambda}\right)}{1-{z_i}^{a}}$$

$$J_{b\lambda} = \frac{\alpha a}{\lambda^2}\sum_{i=1}^{n} \frac{z_i^{a-1} x_i\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)}}{1-{z_i}^{a}}$$

$$J_{\alpha\alpha} = -\frac{n}{\alpha^2} - (a-1)\sum_{i=1}^{n} \frac{w_i}{z_i^2} - a(b-1)$$

$$\sum_{i=1}^{n} \frac{z_i^{a-1} w_i\left((a-1)z_i^{-1} + z_i^a - 1\right)}{\left(1-{z_i}^{a}\right)^2}$$

Where:

$$w_i = \left(1+\frac{x_i}{\lambda}\right)^{-\alpha} \log^2\left(1+\frac{x_i}{\lambda}\right)$$

$$\sum_{i=1}^{n} \frac{\frac{x_i}{\lambda^2}\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)}\left[1+\alpha\log\left(1+\frac{x_i}{\lambda}\right)+\left(1+\frac{x_i}{\lambda}\right)^{-\alpha}\right]}{z_i^2}$$

$$-a(b-1)\sum_{i=1}^{n} \frac{\frac{x_i}{\lambda^2}\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)} z_i^{a-1}\left[\begin{array}{l} z_i^a - 1 + \alpha\log\left(1+\frac{x_i}{\lambda}\right) \\ \left\{\begin{array}{l} 1-\left(1+\frac{x_i}{\lambda}\right)^{-\alpha} \\ z_i^{-1} - z_i^a - z_i^{a-1}\left(1+\frac{x_i}{\lambda}\right)^{-\alpha} \end{array}\right\} \end{array}\right]}{\left(1-{z_i}^{a}\right)^2}$$

and

$$J_{\lambda\lambda} = \frac{n}{\lambda^2} - \frac{(\alpha-1)}{\lambda^3}\sum_{i=1}^{n} \frac{x_i\left(1+\frac{x_i}{\lambda}\right)}{\left(1+\frac{x_i}{\lambda}\right)^2}$$

$$-\alpha(a-1)\sum_{i=1}^{n} \frac{z_i\left[\begin{array}{ll} \frac{(\alpha+1)}{\lambda^4} x_i^2\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+2)} & -\frac{2x_i}{\lambda^3} \\ \left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)} & \end{array}\right] + \frac{\alpha}{\lambda^4} x_i^2\left(1+\frac{x_i}{\lambda}\right)^{-(2\alpha+2)}}{z_i^2}$$

$$+\alpha a(b-1)\sum_{i=1}^{n} \frac{\left(1-{z_i}^{a}\right) Q_i - \alpha a\frac{x_i^2}{\lambda^4} z_i^{2a-2}\left(1+\frac{x_i}{\lambda}\right)^{-(2\alpha+2)}}{\left(1-{z_i}^{a}\right)^2}$$

Table 1: MLEs of the model parameters, the corresponding SEs (given in parentheses) and the statistics AIC, BIC and CAIC

	Estimates				Statistic		
Model	$\hat{\alpha}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\lambda}$	*AIC*	*BIC*	*CAIC*
Kw-GL	11.529	4.368	3.713	5.64	163.616	161.762	163.012
EL	9.875	---	3.981	3.87	197.554	192.18	197.60
L	--	--	4.414	3.456	205.132	267.106	205.850

Where:

$$Q_i = \frac{(\alpha+1)}{\lambda^4} x_i z_i^{a-1}\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+2)} - \frac{2x_i}{\lambda^3} z_i^{a-1}$$

$$\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)} - \alpha(a-1)\frac{x_i^2}{\lambda^4} z_i^{a-2}\left(1+\frac{x_i}{\lambda}\right)^{-(2\alpha+2)}$$

Application: Here, we use a real data set to compare the fits of the Kum-GL distribution and those of other sub-models, i.e. the Exponentiated Lomax (EL) and Lomax distributions. We make a results comparison of the models fit. We consider an uncensored data set corresponding an uncensored data set from consisting of 100 observations on breaking stress of carbon fibers (in Gba): 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.65. These data are used here only for illustrative purposes. The required numerical evaluations are carried out using the Package of Mathcad software.

Tables 1 provide the MLEs of the model parameters. The model selection is carried out using the AIC (Akaike information criterion), the BIC (Bayesian information criterion) and the CAIC (consistent Akaike information criteria):

$$\mathrm{AIC} = -2\ell\left(\hat{\theta}\right) + 2\mathrm{q} \quad \mathrm{BIC} = -2\ell\left(\hat{\theta}\right) + \mathrm{q}\log(\mathrm{n})$$

$$[\mathrm{CAIC} = -2\ell\left(\hat{\theta}\right) + \frac{2\mathrm{qn}}{\mathrm{n}-\mathrm{q}-1}]$$

Where $\ell\left(\hat{\theta}\right)$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, q is the number of parameters and n is the sample size.

Since the values of the AIC, BIC and CAIC are smaller for the Kum-GP distribution compared with those values of the other models, the new distribution seems to be a very competitive model to these data.

Concluding Remarks: The well-known two-parameter Lomax distribution, introduced by Abd-Elfattah *et al.* [12], is extended by introducing two extra shape parameters, thus defining the KW-G exponentiated Pareto (KW-GL) distribution having a broader class of hazard rate and density functions. This is achieved by taking (1) as the baseline cumulative distribution of the generalized class of KW-G distributions defined by Cordeiro *et al.* [7]. A detailed study on the mathematical properties of the new distribution is presented. The new model includes as special sub-models the Pareto, exponentiated Pareto (EP) [13] and Pareto distributions. We obtain the quantile function, skewness and kurtosis. The estimation of the model parameters is approached by maximum likelihood and the observed information matrix is obtained. An application to a real data set indicates that the fit of the new model is superior to the fits of its principal sub-models. We hope that the proposed model may be interesting for a wider range of statistical research.

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