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Stationary Distribution of Markov Matrix and Weights for Determination of Ultimate Cross Efficiency in Dea

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Abstract: This paper firstly reviews the cross efficiency evaluation method which is an extension tool of data envelopment analysis (DEA), then we describe the main shortcomings when the ultimate average cross efficiency scores are used to evaluate and rank the decision making units (DMUs). Subsequently, we eliminate the assumption of average and utilize the stationary distribution of Markov matrix to determine the weights for ultimate cross efficiency scores and the procedures are introduced in detail. In the end, an empirical example is illustrated to examine the validity of the proposed method.

Key words: Data envelopment analysis · Cross-efficiency · Stationary Distribution · Markov matrix · Weights

INTRODUCTION

Data envelopment analysis (DEA) is a methodology for assessing the performances of a group of decision making units (DMUs) that utilize multiple inputs to produce multiple outputs. It measures the performances of the DMUs by maximizing the efficiency of every DMU, respectively, subject to the constraints that none of the efficiencies of the DMUs can be bigger than one. The way to measure the optimistic efficiencies of the DMUs is referred to as self-evaluation. Basically, DEA provides a categorical classification of the units into efficient and inefficient ones [1]. However, although DEA is strong in identifying the inefficient units it is weak in discriminating among the efficient units. The cross efficiency method was developed as a DEA extension technique that could be utilized to identify efficient DMUs and to rank DMUs using cross efficiency scores that are linked to all DMUs [2]. The basic idea of the cross efficiency evaluation is to evaluate the overall efficiencies of the DMUs through both self- and peer-evaluations. The self-evaluation allows each DMU to be evaluated with the most favorable input and output weights so that the best relative efficiency can be achieved for each DMU, whereas the peer-evaluation requests each DMU to be evaluated with the weights determined by the other DMUs. The overall efficiency of a DMU is the average of its self-evaluation efficiency and peer-evaluation efficiencies.

Although average cross efficiency has been widely used, there are still several disadvantages for utilizing the final average cross efficiency to evaluate and rank DMUs, like the losing association with the weights by averaging among the cross efficiencies [3]. Considering the shortcomings above, Wu, Liang and Yang (2009) [4] eliminate the average assumption and use the Shapley values to determine the ultimate cross efficiency scores. Jie Wu *et al*. (2011) [5] utilize the Shannon entropy to determine the weights for ultimate cross efficiency scores. In the current paper, we will propose an approach based on stationary distribution of Markov matrix instead of calculating the average cross efficiency scores. This approach has several advantages. Firstly, in this method, the most productive scale size (MPSS) units [6] get the best rank and the interior points of the smallest production possibility sets (PPSs) which are inefficient in all models lie at the end of the ranking list [7]. Secondly, this method fully takes advantage of the global information in the matrix rather than row information in Shannon entropy method.

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Table 1: A generalized cross efficiency matrix.

	Rated DMU_j			
Rating DMU_d	1	2	…	N
1	E_{11}	E_{12}	…	E_{1n}
2	E_{21}	E_{22}	…	E_{2n}
…	…	…	…	…
n	E_{n1}	E_{n1}	…	E_{nn}
Mean	$\bar{E}_1$	$\bar{E}_2$	…	$\bar{E}_n$

The rest of the paper is organized as follows. Section 2 briefly reviews the cross efficiency evaluation model. Section 3 proposes the weights determination based on stationary distribution of Markov matrix and ultimate cross efficiency score. Section 4 examines a numerical example to show the effectiveness of the proposed weights determination approach. The paper concludes in Section 5.

Cross-Efficiency Evaluation: Using the traditional denotations in DEA, we assume that there are a set of n DMUs and each $DMU_j(j = 1,2,...,n)$ produces s different outputs using m different inputs which are denoted as $x_{ij}(i = 1,2,...,m)$ and $y_{rj}(r = 1,2,...s)$, respectively. For any evaluated $DMU_d(d = 1,2,...,n)$, the efficiency score E_{dd} can be calculated by using the following CCR model.

$$\begin{aligned} &\max \sum_{r=1}^{s} \mu_{rd} y_{rd} = E_{dd} \\ &s.t. \quad \sum_{i=1}^{m} \omega_{id} x_{ij} - \sum_{r=1}^{s} \mu_{rd} y_{ij} \ge 0, \quad j = 1,2,\cdots,n \\ &\qquad \sum_{i=1}^{m} \omega_{id} x_{id} = 1 \\ &\qquad \omega_{id} \ge 0, \quad i = 1,2,\cdots,m \\ &\qquad \mu_{rd} \ge 0, \quad r = 1,2,\cdots,s. \end{aligned} \tag{1}$$

For each $DMU_d(d = 1,2,...,n)$, we can obtain a group of optimal input weights $\omega_{1d}^*, \omega_{2d}^*, \cdots, \omega_{md}^*$ and a group of optimal output weights $\mu_{1d}^*, \mu_{2d}^*, \cdots, \mu_{sd}^*$ by solving the above model (1) and the cross-efficiency of each DMU_j), using the weights of DMU_d, namely E_{dj}, can be calculated as follows.

$$E_{dj} = \frac{\sum_{r=1}^{s} \mu_{rd}^* y_{rj}}{\sum_{i=1}^{m} \omega_{id}^* x_{ij}}, \quad d, j = 1,2,\cdots,n \tag{2}$$

As shown in the Table 1 of cross efficiency matrix (CEM), for each column, E_{dj} is the cross efficiency score of DMU_j using the weights that $DMU_d(d = 1,2,...,n)$, has chosen. We can also find that the elements in the diagonal are the special cases that can be seen as self-evaluation.

For each $DMU_j(j = 1,2,...,n)$, the average of all $E_{dj}(d = 1,2,...,n)$, namely, $\bar{E}_j = \frac{1}{n}\sum_{d=1}^{n} E_{dj}$ $(j = 1,2,...,n)$ can be treated as a new efficiency measure, that is, the cross efficiency score (CES) for DMU_j.

Stationary Distribution of Markov Matrix and Ultimate CES Determination: Notice that the CEM is a square matrix with entries whose real values are the performance evaluation for some DMUs by others. If we normalize the real values in each row, the modified CEM will become an analogue of Markov matrix. So we can use the stationary distribution of Markov matrix, which contains the information of the matrix, to determine the ultimate CES of each DMU.

Stationary Distribution of Markov Matrix: For a finite-state homogeneous Markov chain $\{X_t, t = 0,1,2,...,...\}$, let P denote the matrix of one-step transition probabilities $P_{ij}(i = 1,...,n; j = 1,...,n)$, namely Markov matrix, so that

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

If $\pi_j = \lim_{n\to\infty} P_{ij}^n > 0$ $(j = 1,2,...n)$ or

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \\ \vdots & \vdots & \cdots & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}$$

where, P_{ij}^n is the n-step transition probabilities? In this case, $\pi = \{\pi_j, j = 1,2,...,n\}$ is a stationary distribution and there is no other stationary distribution.

Obviously, stationary distribution is the limit of Markov matrix. For an irreducible Markov chain, the stationary distribution π of Markov chain exists if and only if all states are recurrent [8].

Determination Ultimate CES Using Weights from Stationary Distribution: According to the concept of stationary distribution of Markov matrix, it can be used to evaluate the DMUs. Assume we have obtained the CEM as shown in Table 1. The steps for determining the weights of DMUs and determination of the ultimate efficiency scores are as follows.

Step 1. Row Normalization: In a row-normalized matrix, the *(d, j)* th element of CEM becomes $\tilde{E}_{dj} = E_{dj} / Z_d$, where Z_d is the sum of the dth row of CEM. After row normalization, each row of normalized CEM will sum to one.

In the classical CCR model, it assumes that $x_{ij} \geq 0$ and $y_{rj} \geq 0$ and further assumes that each DMU has at least one positive input and one positive output value. In a real-life scenario, positive inputs and positive outputs value are quite common. At the same time, model (1) require $\omega_{id} \geq 0$ and $\mu_{rd} \geq 0$. A fully rigorous development would replace $\omega_{id} \geq 0$ and $\mu_{rd} \geq 0$ with $\omega_{id} \geq \varepsilon > 0$ and $\mu_{rd} \geq \varepsilon > 0$, where ε is a non-Archimedean element smaller than any positive real number [9]. This condition guarantees that solutions will be positive in these variables, then the efficiency score E_{dj} will also be positive. The normalized CEM will be Markov matrix with recurrent states. So, the stationary distribution exists in most cases.

Step 2. Computing π Algebraically: If the normalized CEM is obtained, then we can solve the set of following equations to get the unique solution π [10].

$$\begin{cases} \pi = \pi P \\ \sum_{d=1}^{n} \pi_d = 1 \end{cases} \tag{3}$$

where, *P* is the normalized CEM.

Step 3. Determination of Ultimate Cross Efficiency: Finally, the weights, i.e. the value of $\pi = \{\pi_1, \pi_2, \ldots \pi_n\}$, can be used to determine the ultimate cross efficiency and the ultimate cross efficiency of each DMU is expressed as follows:

$$E_j = \sum_{d=1}^{n} \pi_d \tilde{E}_{dj}, \qquad j = 1, 2, \cdots, n \tag{4}$$

Illustration: In order to illustrate the method which has been proposed above, we consider a simple numerical example shown in the Table 2. There are five DMUs, each DMU has three inputs X_1, X_2, X_3 and two outputs Y_1, Y_2 After solving the CCR model (1), we can obtain the cross efficiency matrix listed in Table 3 according to formula (2). The row-normalized CEM is shown in Table 4.

Table 2: Numerical example

	X_1	X_2	X_3	Y_1	Y_2
DMU_1	7	7	7	4	4
DMU_2	5	9	7	7	7
DMU_3	4	6	5	5	7
DMU_4	5	9	8	6	2
DMU_5	6	8	5	3	6

Table 3: Cross efficiency matrix

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
DMU_1	0.6578	0.4478	0.3710	0.4587	0.4082
DMU_2	0.9333	1.0000	0.7489	1.0000	0.7143
DMU_3	1.0000	0.9965	1.0000	0.9313	1.0000
DMU_4	0.8000	0.7323	0.2092	0.8571	0.1786
DMU_5	0.4500	0.4643	0.6402	0.3817	0.8571

Table 4: Row-normalized Cross efficiency matrix

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
DMU_1	0.2807	0.1911	0.1583	0.1957	0.1742
DMU_2	0.2123	0.2275	0.1703	0.2275	0.1625
DMU_3	0.2029	0.2022	0.2029	0.1890	0.2029
DMU_4	0.2881	0.2637	0.0753	0.3086	0.0643
DMU_5	0.1611	0.1662	0.2292	0.1366	0.3069

Then the stationary distribution value of row-normalized CEM can be obtained after solving the formula (3) and the values are:

$$\pi = \{\pi_1, \pi_2, \cdots \pi_n\} = \{0.1384, 0.2555, 0.2926, 0.1510, 0.1625\}$$

After we can get the weights based on the stationary distribution of Markov matrix, the ultimate cross efficiency scores of each DMU via formula (4) are

$$E_1 = 0.4496,\ E_2 = 0.5689,\ E_3 = 0.4035,\ E_4 = 0.5175,\ E_5 = 0.5652.$$

Instead of the average cross efficiency, we determine the ultimate cross efficiency using the stationary distribution of matrix in Markov chain theory. Stationary distribution of matrix is a feasible and effective method of objective weighting, it fully exploits the information of the data itself and more conform to the objective reality. So each DMU will have more motivation to accept this result of the efficiency [11-13].

CONCLUSIONS

There are some disadvantages for utilizing the ultimate average cross efficiency to evaluate and rank DMUs. Aiming at the flaws, we eliminate the assumption of average and utilize the concept of stationary distribution of Markov matrix to determine the ultimate cross efficiency scores for each DMU. Finally, a numerical example is illustrated to prove the effectiveness of the proposed approach. We should point out that the numerical example in this paper is chosen only for illustrative purposes and for better understanding of the main principles of the proposed approach, so the utilization of our proposed methods in more real-world cases and contexts would obviously be interesting in future research.

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