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# Solution of Fourth Order Singularly Perturbed Boundary Value Problem Using Septic Spline

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**Abstract:** In this paper the fourth order singularly perturbed boundary value problem is solved using septic spline. The given method is proved to be fourth order convergent. To illustrate the efficiency of the method two examples are considered. The method is also compared with the existing method and it is evident that the method is better than the existing one.

**MSC:** 65L10
**Key words:** Singular perturbation · Septic spline · Fourth-order ordinary differential equation · Boundary layer · Self adjoint

## INTRODUCTION

Perturbation theory is a well-known and important theory in these days. For two basic reasons singular perturbed problems have gained importance. Firstly, they appear in many areas of science and engineering, for instance fluid mechanics, combustion, nuclear engineering, elasticity, quantum mechanics, chemical reactor theory, convention-diffusion process, control theory, *etc*. A few good examples are the modelling of steady and unsteady viscous flow problems with large Reynolds number, WKB Theory, boundary layer problems and convective-heat transport problems with large Peclet number.

Secondly, the formation of sharp boundary layers in numerical methods when ε, the coefficient of highest derivative, approaches to zero creates problem. Both the analytical and numerical handling of these problems is becoming interesting for researchers. Since, in general, the classical numerical methods fail to produce good approximations for these equations. Hence one has to search for the non-classical methods. For analytical discussion on singular perturbation problems, one can refer to, Kevorkian and Cole [1], Bender and Orsazag [2], Mally [3], Nayfeh [4, 5], Van Dyke [6]. From last 20 years a large number of articles have been appearing on non-classical methods, with mostly second order equations such as [7-11]. Only few researchers have developed higher order singularly perturbed problems such as [10, 12-14]. A survey article by Patidar and Kadalbajoo [15] is considerable in this respect.

The solution of singularly perturbed boundary value problems is described by slowly and rapidly varying parts. So there are thin transition layers where the solution can jump suddenly, while away from the layers the solution varies slowly and behaves regularly. Ghazala [16] solved the third order singularly perturbed boundary value problem using quartic spline and the method is proved to be second order convergent.

Ghazala and Nadia [17] solved the fourth order singularly perturbed boundary value problem using quintic spline and the method is proved to be second order convergent.

There are three standard approaches to solve singularly perturbed boundary value problems numerically, the finite difference method [14, 18, 19], the finite element method [20] and spline approximation methods [9, 10, 11]. In the present paper the third technique, *i.e.*, spline approximation method has been used to solve singularly perturbed self adjoint boundary value problem arising in the study of chemical reactor theory, of the form:

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$$\left.\begin{aligned} &Lu(x) = -\varepsilon u^{(4)}(x) + p(x)u(x) = f(x),\ p(x) \geq p \geq 0,\ a \leq x \leq b, \\ &u(a) = \alpha_0,\ u(b) = \alpha_1,\ u^{(1)}(a) = \alpha_2,\ u^{(1)}(b) = \alpha_3, \end{aligned}\right\} \tag{1.1}$$

or

$$\left.\begin{aligned} &Lu(x) = -\varepsilon u^{(4)}(x) + p(x)u(x) = f(x),\ p(x) \geq p \geq 0,\ a \leq x \leq b, \\ &u(a) = \alpha_0,\ u(b) = \alpha_1,\ u^{(2)}(a) = \alpha_4,\ u^{(2)}(b) = \alpha_5, \end{aligned}\right\} \tag{1.2}$$

where $\alpha_i$, $i=0, 1, 2, \dots, 5$ are finite real constants and $\varepsilon$, is a small positive parameter ($0 < \varepsilon \leq 1$). Further functions $f(x)$ and $p(x)$ are smooth functions and $p(x) = p =$ constant. It is known that the most classical methods fail when $\varepsilon$ is small relative to the mesh width $h$. Our target is to develop a method to give accurate numerical approximation of (1.1) when $\varepsilon$ is either small or large as compared to $h$.

This paper is organized in five sections. In Section 2, the consistency relations in terms of values of spline and its six derivatives at knots are determined using derivatives continuities at knots. Consistent end conditions are determined in Section 3. In Section 4, it is proved that the septic spline solution for the fourth order singularly perturbed differential equation is of $O(h^4)$. In Section 5, two examples are considered to show the accuracy of the method developed.

**Septic Spline and its Consistency Relations:** To develop the consistency relations the following seventh degree spline is considered:

$$\begin{aligned} S_i(x) = {} & a_i(x-x_i)^7 + b_i(x-x_i)^6 + c_i(x-x_i)^5 + d_i(x-x_i)^4 + e_i(x-x_i)^3 \\ & + q_i(x-x_i)^2 + g_i(x-xi) + l_i \end{aligned} \tag{2.1}$$

defined on $[a, b]$, where $x \in [x_i, x_{i+1}]$ with equally spaced knots, $x_i = a + ih$, $i = 0,1, 2,\dots N$,

$h = (b-a)/N$ and $S(x) \in C^6[a, b]$.

To determine the eight coefficients introduced in Eq. (2.1), the eight conditions are required. These conditions can be defined in many ways such as in terms of second, fourth and sixth derivatives at both ends of each subinterval.

Let

$$\left.\begin{aligned} &S_i(x_i) = u_i, && S_i(x_{i+1}) = u_{i+1}, \\ &S_i^{(2)}(x_i) = m_i, && S_i^{(2)}(x_{i+1}) = m_{i+1}, \\ &S_i^{(4)}(x_i) = M_i, && S_i^{(4)}(x_{i+1}) = M_{i+1}, \\ &S_i^{(6)}(x_i) = F_i, && S_i^{(6)}(x_{i+1}) = F_{i+1}, \end{aligned}\right\}$$

The coefficients determined are as follows:

$$a_i = \frac{F_{i+1} - F_i}{5040h},$$
$$b_i = \frac{F_i}{720},$$
$$c_i = \frac{-hF_i}{360} - \frac{hF_{i+1}}{720} - \frac{M_i}{120h} + \frac{M_{i+1}}{120h},$$
$$d_i = \frac{M_i}{24},$$
$$e_i = \frac{h^3F_i}{270} + \frac{7h^3F_{i+1}}{2160} - \frac{m_i}{6h} + \frac{m_{i+1}}{6h} - \frac{hM_i}{18} - \frac{hM_{i+1}}{36},$$
$$q_i = \frac{m_i}{2},$$
$$g_i = \frac{-2h^5F_i}{945} - \frac{31h^5F_{i+1}}{15120} - \frac{hm_i}{3} - \frac{hm_{i+1}}{6} + \frac{h^3M_i}{45} + \frac{7h^3M_{i+1}}{360} - \frac{u_i}{h} + \frac{u_{i+1}}{h},$$
$$l_i = u_i.$$

From the continuity of the first, third and fifth derivative at the point $x = x_i$ the following relations are derived

$$\frac{31h^5F_{i-1}}{15120} + \frac{4h^5F_i}{945} + \frac{31h^5F_{i+1}}{15120} + \frac{hm_{i-1}}{6} + \frac{2hm_i}{3} + \frac{hm_{i+1}}{6} - \frac{7h^3M_{i-1}}{360} - \frac{2h^3M_i}{45} - \frac{7h^3M_{i+1}}{360} - \frac{u_{i-1}}{h} + \frac{2u_i}{h} - \frac{u_{i+1}}{h} = 0 \quad (2.2)$$

$$7h^4F_{i-1} + 16h^4F_i + 7h^4F_{i+1} + 360m_{i-1} - 720m_i + 360m_{i+1} - 60h^2M_{i-1} - 240h^2M_i - 60h^2M_{i+1} = 0 \quad (2.3)$$

$$h^2F_{i-1} + 4h^2F_i + h^2F_{i+1} - 6M_{i-1} + 12M_i - 6M_{i+1} = 0 \quad (2.4)$$

which leads to the following consistency relation in terms of $M_i$ and $u_i$

$$h^4M_{i-3} + 120h^4M_{i-2} + 1191h^4M_{i-1} + 2416h^4M_i + 1191h^4M_{i+1} + 120h^4M_{i+2} + h^4M_{i+3} - 840u_{i-3} + 7560u_{i-1} - 13440u_i + 7560u_{i+1} - 840u_{i+3} = 0, \quad i = 3,4,\ldots,N-3. \quad (2.5)$$

Using Eq. (1.1), the Eq. (2.5) can be written as

$$\left(ph^4 - 840\varepsilon\right)u_{i-3} + 120ph^4u_{i-2} + \left(1191ph^4 + 7560\varepsilon\right)u_{i-1} + \left(2416ph^4 - 13440\varepsilon\right)u_i + \left(1191h^4 + 7560\varepsilon\right)u_{i+1} + 120ph^4u_{i+2} + \left(ph^4 - 840\varepsilon\right)u_{i+3} - h^4\left(f_{i-3} + 120f_{i-2} + 1191f_{i-1} + 2416f_i + 1191f_{i+1} + 120f_{i+2} + f_{i+3}\right) = 0, \quad i = 3,4,\ldots,N-3. \quad (2.6)$$

**End Conditions:** Since the system (2.6) consists of ($N$ - 5) equations in ($N$ -1) unknowns, so four more equations are required, as the end conditions. Consider the end conditions for the system (1.1), in the following form

$$\sum_{l=0}^{6} a_{k+l}M_{k+l} = \frac{1}{h^4}\left[\sum_{j=k}^{k+5} b_ju_j + hc_0u_0^{(1)}\right], \; k = 0,1,N-3,N-2, \quad (3.1)$$

where all the coefficients $a_is$, $i = 0, 1,\dots 6$, $b_is$; $i = 0,1,\dots,5$ and $c_0$ are to be determined using the method of undetermined coefficients. The value of coefficients for $k = 0$ can be calculated, as

$$a_0 = 1, \quad a_1 = -\frac{84847}{1280}, \quad a_2 = \frac{5312349}{5440}, \quad a_3 = \frac{42625923}{10880},$$
$$a_4 = \frac{5830349}{5440}, \quad a_5 = -\frac{444639}{21760}, \quad a_6 = 0, \quad b_0 = \frac{54915}{544},$$
$$b_1 = \frac{1685061}{272}, \quad b_2 = -\frac{3371697}{136}, \quad b_3 = \frac{5000583}{136}, \quad b_4 = -\frac{778239}{32},$$
$$b_5 = \frac{1644741}{272}, \quad c_0 = \frac{17325}{136}.$$

Substituting the values of $a_ms$, $b_ns$ for $m = 0,1,\dots,6$, $n = 0,1,\dots,5$ and $c_0$ in Eq. (3.1) the required end condition for $i = 1$ is determined, as

$$\begin{aligned}
&-(1442399ph^4 + 134804880\varepsilon)u_1 + (21249396ph^4 + 539471520\varepsilon)u_2 + (85251846ph^4 \\
&-800093280\varepsilon)u_3 + (23321396ph^4 + 529202520\varepsilon)u_4 - (444639ph^4 + 131579280\varepsilon)u_5 \\
&+43520ph^4u_6 - h^4(-1442399f_1 + 21249396f_2 + 85251846f_3 + 23321396f_4 \\
&-444639f_5 + 43520f_6) \\
&= 2772000h\alpha_2\varepsilon - (21760ph^4 - 2196600\varepsilon)\alpha_0 + 21760h^4f_0 + O(h^8).
\end{aligned} \tag{3.2}$$

Again, using the Taylor's series for the Eq. (3.1), the values of coefficients for $k = 1$ can be calculated, as

$$a_1 = 1, \quad a_2 = -\frac{248348494}{15737335}, \quad a_3 = \frac{17249030576}{15737335}, \quad a_4 = \frac{62123692776}{15737335},$$
$$a_5 = \frac{16858368451}{15737335}, \quad a_6 = -\frac{320824034}{15737335}, \quad a_7 = 2, \quad b_1 = \frac{145913040}{3147467},$$
$$b_2 = \frac{18491263056}{3147467}, \quad b_3 = -\frac{75355806624}{3147467}, \quad b_4 = \frac{113691203136}{3147467}, \quad b_5 = -\frac{76004059824}{3147467},$$
$$b_6 = \frac{19031487216}{3147467}, \quad c_0 = \frac{5544000}{3147467}.$$

Substituting the values of $a_ms$, $b_ns$ for $m = 1, 2,\dots,7$, $n = 1,2,\dots,6$ and $c_0$ in Eq. (3.1) the required end condition for $i = 2$ is determined, as

$$\begin{aligned}
&(15737335ph^4 - 729565200\varepsilon)u_1 - (248348494ph^4 + 92456315280\varepsilon)u_2 \\
&+(17249030576ph^4 + 376779033120\varepsilon)u_3 + (62123692776ph^4 - 568456015680\varepsilon)u_4 \\
&+(16858368451ph^4 + 380020299120\varepsilon)u_5 - (320824034ph^4 + 95157436080)u_6 \\
&+31474670ph^4u_7 - h^4(15737335f_1 - 248348494f_2 + 17249030576f_3 + 62123692776f_4 \\
&+16858368451f_5 - 320824034f_6 + 31474670f_7) \\
&= 27720000h\alpha_2\varepsilon + O(h^8).
\end{aligned} \tag{3.3}$$

Similarly, the end condition for $i = N - 2$ is:

$$\begin{aligned}&31474670ph^4u_{N-7}-(320824034ph^4+95157436080)u_{N-6}+(16858368451ph^4\\&+380020299120\varepsilon)u_{N-5}+(62123692776ph^4-568456015680\varepsilon)u_{N-4}\\&+(17249030576ph^4+376779033120\varepsilon)u_{N-3}-(248348494ph^4+92456315280\varepsilon)u_{N-2}\\&+(15737335ph^4-729565200\varepsilon)u_{N-1}-h^4(31474670f_{N-7}-320824034f_{N-6}\\&+16858368451f_{N-5}+62123692776f_{N-4}+17249030576f_{N-3}-248348494f_{N-2}\\&+15737335f_{N-1})\\&=-27720000h\alpha_3\varepsilon+O(h^8)\end{aligned} \tag{3.4}$$

and for $i = N - 1$, the end condition is:

$$\begin{aligned}&43520ph^4u_{N-6}-(444639ph^4+131579280\varepsilon)u_{N-5}+(23321396ph^4+529202520\varepsilon)u_{N-4}\\&+(85251846ph^4-800093280\varepsilon)u_{N-3}+(21249396ph^4+539471520\varepsilon)u_{N-2}-(1442399ph^4\\&+134804880\varepsilon)u_{N-1}-h^4(43520f_{N-6}-444639f_{N-5}+23321396f_{N-4}+85251846f_{N-3}\\&+21249396f_{N-2}-1442399f_{N-1})\\&=-2772000h\alpha_3\varepsilon-(21760ph^4-2196600\varepsilon)\alpha_1+21760h^4f_N+O(h^8).\end{aligned} \tag{3.5}$$

The end conditions for the solution of the system (1.2) can be calculated in the same manner and are given as follows: for $i = 1$

$$\begin{aligned}&-(25554553ph^4+2865285360\varepsilon)u_1+(424736812ph^4+10931185440\varepsilon)u_2\\&+(1710601962ph^4-16109120160\varepsilon)u_3+(468560812ph^4+10635967440\varepsilon)u_4\\&-(8963833ph^4+2642718960\varepsilon)u_5+877440ph^4u_6-h^4(-25554553f_1\\&+424736812f_2+1710601962f_3+468560812f_4-8963833f_5+877440f_6)\\&=-(438720ph^4+49971600\varepsilon)\alpha_0+438720f_0h^4-7560000h^2\alpha_4\varepsilon,\end{aligned} \tag{3.6}$$

for $i = 2$

$$\begin{aligned}&(6521255ph^4-724323600\varepsilon)u_1-(11796622ph^4+36583361640\varepsilon)u_2\\&+(7455263088ph^4+153557782560\varepsilon)u_3+(25847327688ph^4-233933721840\varepsilon)u_4\\&+(6989950963ph^4+157149160560\varepsilon)u_5-(132814242ph^4+39465536040\varepsilon)u_6\\&+13042510ph^4u_7-h^4(6521255f_1-11796622f_2+7455263088f_3+25847327688f_4\\&+6989950963f_5-132814242f_6+13042510f_7)\\&=-3780000h^2\alpha_4\varepsilon,\end{aligned} \tag{3.7}$$

for $i = N - 2$

$$\begin{aligned}&13042510ph^4u_{N-7}-(132814242ph^4+39465536040\varepsilon)u_{N-6}+(6989950963ph^4\\&+157149160560\varepsilon)u_{N-5}+(25847327688ph^4-233933721840\varepsilon)u_{N-4}\\&+(7455263088ph^4+153557782560\varepsilon)u_{N-3}-(11796622ph^4+36583361640\varepsilon)u_{N-2}\\&+(6521255ph^4-724323600\varepsilon)u_{N-1}-h^4(13042510f_{N-7}-132814242f_{N-6}\\&+6989950963f_{N-5}+25847327688f_{N-4}+7455263088f_{N-3}-11796622f_{N-2}+6521255f_{N-1}\\&=-3780000\alpha_5h^2\end{aligned} \tag{3.8}$$

and for $i = N - 1$

$$\begin{aligned}
&877440ph^4u_{N-6} - (8963833ph^4 + 2642718960\varepsilon)u_{N-5} + (468560812ph^4 \\
&+ 10635967440\varepsilon)u_{N-4} + (1710601962ph^4 - 16109120160\varepsilon)u_{N-3} \\
&+ (424736812ph^4 + 10931185440\varepsilon)u_{N-2} - (25554553ph^4 + 2865285360\varepsilon)u_{N-1} \\
&- h^4(877440f_{N-6} - 8963833f_{N-5} + 468560812f_{N-4} + 1710601962f_{N-3} \\
&+ 424736812f_{N-2} - 25554553f_{N-1}) \\
&= -7560000\alpha_5h^2\varepsilon - (438720ph^4 + 49971600\varepsilon)\alpha_1 + 438720h^4f_N.
\end{aligned} \tag{3.9}$$

**Convergence of the Method:** The system of Eqns. (3.2), (3.3), (2.6), (3.4) and (3.5), provides the required solution of BVP (1.1) which can be written in matrix form, as

$$AU - h^4DF = C, \tag{4.1}$$

where $U = u_i$, $C = c_i$ and $F = f_i$ are the $(N-1)$ dimensional column vectors. $A$ and $D$ are $(N-1)\times(N-1)$ matrices, where $A = a_{ij}$ and $a_{ij}s$ are the coefficients of $u_j$ and

$$D = [D1\ D2] \tag{4.2}$$

$$D_1 = \begin{bmatrix} -1442399 & 21249396 & 85251846 & 23321396 & -444639 \\ 15737335 & -248348494 & 17249030576 & 62123692776 & 16858368451 \\ 120 & 1191 & 2416 & 1191 & 120 \\ & \ddots & \ddots & \ddots & \\ & & & 1 & 120 \\ & & & 0 & 1 \\ & & & 31474670 & -320824034 \\ & & & 0 & 43520 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 43520 & 0 & & & \\ -320824034 & 31474670 & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & \ddots & \\ 1191 & 2416 & 1191 & 120 & 1 \\ 120 & 1191 & 2416 & 1191 & 120 \\ 16858368451 & 62123692776 & 17249030576 & -248348494 & 15737335 \\ -444639 & 23321396 & 85251846 & 21249396 & -1442399 \end{bmatrix}$$

Also,

$c_1 = 21760h^4f_0 - (21760ph^4 - 2196600\varepsilon)\alpha_0 + 2772000h\alpha_2\varepsilon,$

$c_2 = 277720000\alpha_2h\varepsilon,$

$c_3 = h^4f_0 - (ph^4 - 840\varepsilon)\alpha_0,$

$c_i = 0, \qquad i = 4,5,\ldots,N-4,$

$c_{N-3} = h^4f_N - (ph^4 - 840\varepsilon)\alpha_1,$

$c_{N-2} = 277720000\alpha_3h\varepsilon,$

$c_{N-1} = 21760h^4f_N - (21760ph^4 - 2196600\varepsilon)\alpha_1 + 2772000h\alpha_3\varepsilon.$

Let $\bar{U}$ be the exact solution of BVP (1.1) and $U$ be the approximate solution then Eq. (4.1) can be rewritten as,

$$A\,\overline{U} - h^4 DF = T(h) + C, \tag{4.3}$$

where

$$\overline{U} = (u(x_1), u(x_2), \ldots, u(x_{N-1}))^T$$

and

$$T(h) = (t_1(h), t_2(h), \ldots, t_{N-1}(h))^T,$$

while $T(h)$ denotes the truncation error and $t_i(h)$ are calculated, as

$$\left.\begin{aligned}
t_1(h) &= \frac{1056377}{3133440}\varepsilon h^8 u^{(12)}(\xi_1), && x_0 \le \xi_1 \le x_6,\\
t_2(h) &= \frac{259927729}{755392080}\varepsilon h^8 u^{(12)}(\xi_2), && x_1 \le \xi_2 \le x_7,\\
t_i(h) &= 7\varepsilon h^8 u^{(12)}(\xi_i), && x_{i-3} \le \xi_i \le x_{i+3}, \quad i = 3,4,\ldots,N-3,\\
t_{N-2}(h) &= \frac{259927729}{755392080}\varepsilon h^8 u^{(12)}(\xi_{N-2}), && x_{N-7} \le \xi_{N-2} \le x_{N-1},\\
t_{N-1}(h) &= \frac{1056377}{3133440}\varepsilon h^8 u^{(12)}(\xi_{N-1}), && x_{N-6} \le \xi_{N-1} \le x_N.
\end{aligned}\right\} \tag{4.4}$$

From Eq. (4.1) and Eq. (4.3), it follows that:

$$A(\overline{U} - U) = AE = T(h), \tag{4.5}$$

where

$$E = \overline{U} - U = (e_1, e_2, \ldots, e_{N-2}, e_{N-1})^T.$$

To determine the error bound, the row sums $S_1, S_2, \ldots, S_{N-2}, S_{N-1}$ of matrix $A$ are calculated as

$$\left.\begin{aligned}
S_1 &= \sum_j a_{1,j} = 667450640\,ph^4 + 2196600\varepsilon,\\
S_2 &= \sum_j a_{2,j} = 95709131280\,ph^4,\\
S_3 &= \sum_j a_{3,j} = 5039\,ph^4 + 840\varepsilon,\\
S_i &= \sum_j a_{i,j} = 5040\,ph^4, && i = 4,5,\ldots,N-4,\\
S_{N-3} &= \sum_j a_{N-3,j} = 5039\,ph^4 + 840\varepsilon,\\
S_{N-2} &= \sum_j a_{N-2,j} = 95709131280\,ph^4,\\
S_{N-1} &= \sum_j a_{N-1,j} = 667450640\,ph^4 + 2196600\varepsilon.
\end{aligned}\right\} \tag{4.6}$$

Since the matrix $A$ is observed to be irreducible and monotone. It follows that, $A^{-1}$ exists and its elements are non-negative. Using this result the Eq. (4.5) can be written as

$$E = A^{-1}T(h). \tag{4.7}$$

Also, from the theory of matrices it can be written as

$$\sum_{i=1}^{N-1} a_{k,i}^{-1} S_i = 1, \qquad k = 1,2,\ldots,N-1, \tag{4.8}$$

where $a_{k,i}^{-1}$ is the $(k, i)th$ element of the matrix $A^{-1}$.

From Eq. (4.6), it follows that

$$\sum_{i=1}^{N-1} a_{k,i}^{-1} \le 1/\min S_i = 1/(h^4 B_{i0}), \tag{4.9}$$

where

$$B_{i0} = (1/h^4)\min S_i > 0, \tag{4.10}$$

for some $i_0$ between 1 and $N-1$.

From Eq. (4.7), it can be written as

$$e_k = \sum_{i=1}^{N-1} a_{k,i}^{-1} T_i(h) \qquad k = 1,2,\ldots,N-1. \tag{4.11}$$

Using Eq. (4.4) in Eq. (4.11) the following result is obtained,

$$|e_k| \le \frac{lh^4}{B_{i0}}, \qquad k = 1,2,\ldots,N-1, \tag{4.12}$$

where $l$ is a constant independent of $h$. From Eq. (4.12) it follows that,

$$\|E\| = O(h^4). \tag{4.13}$$

Similarly, the method developed for the system of Eqns. (3.6), (3.7), (2.6), (3.8) and (3.9), preserves fourth order convergence. The results can be summarized in the following theorems

**Theorem 4.1:** The method given by system (3.2), (3.3), (2.6), (3.4) and (3.5) for solving the boundary value problem (1.1) for sufficiently small $h$ gives a fourth order convergent solution.

**Theorem 4.2:** The method given by system (3.6), (3.7), (2.6), (3.8) and (3.9) for solving the boundary value problem (1.2) for sufficiently small $h$ gives a fourth order convergent solution.

**Numerical Results**

**Example1:** For $x \in [0, 1]$, consider the differential system:

$$\begin{aligned}-\varepsilon u^{(4)}(x) + pu(x) = {} & (x-1)^4 x^8 \sin(\varepsilon x) - \varepsilon x^4(-16\varepsilon^3 (x-1)^3 x^3 (3x-2)\cos(\varepsilon x) \\ & + 96\varepsilon x(14 - 84x + 180x^2 - 165x^3 + 55x^4)\cos(\varepsilon x) \\ & + \varepsilon^4 (x-1)^4 x^4 \sin(\varepsilon x) - 24\varepsilon^2 (x-1)^2 x^2 (14 - 44x + 33x^2)\sin(\varepsilon x) \\ & + 24(70 - 504x + 1260x^2 - 1320x^3 + 495x^4)\sin(\varepsilon x)),\end{aligned}$$

with $\quad u(0) = 0, u(1) = 0, u^{(1)}(0) = 0, u^{(1)}(1) = 0.$

The exact solution of the above system is,

Table 1: The results developed by the method

| ε | h=1/16 | h=1/32 | h=1/64 | h=1/128 |
|---|---|---|---|---|
| 1/16 | 1.666 e-006 | 1.31 e-007 | 2.614 e-009 | 6.716 e-011 |
| 1/32 | 8.537 e-007 | 6.736 e-008 | 1.344 e-009 | 3.452 e-011 |
| 1/64 | 4.520 e-007 | 3.569 e-008 | 7.128 e-010 | 1.829 e-011 |
| 1/128 | 2.60 e-007 | 2.049 e-008 | 4.092 e-010 | 1.05 e-011 |

Table 2: The results developed by Ghazala and Nadia [17]

| ε | h=1/16 | h=1/32 | h=1/64 | h=1/128 |
|---|---|---|---|---|
| 1/16 | 1.315 e-006 | 1.617 e-007 | 2.853 e-008 | 6.682 e-009 |
| 1/32 | 6.703 e-007 | 8.170 e-008 | 1.434 e-008 | 3.355 e-009 |
| 1/64 | 3.489 e-007 | 4.177 e-008 | 7.249 e-009 | 1.692 e-009 |
| 1/128 | 1.915 e-007 | 2.201 e-008 | 3.717 e-009 | 8.619 e-010 |

Table 3: The results developed by the method

| ε | h=1/10 | h=1/20 | h=1/40 | h=1/80 |
|---|---|---|---|---|
| 1/16 | 8.6 e-003 | 1.506 e-004 | 2.951 e-006 | 8.053 e-008 |
| 1/32 | 2.5 e-003 | 4.772 e-005 | 9.202 e-007 | 2.246 e-008 |
| 1/64 | 1.7 e-003 | 3.397 e-005 | 6.468 e-007 | 1.425 e-008 |
| 1/128 | 8.653 e-004 | 1.792 e-005 | 3.387 e-007 | 7.199 e-009 |

Table 4: The results developed by Ghazala and Nadia [17]

| ε | h=1/10 | h=1/20 | h=1/40 | h=1/80 |
|---|---|---|---|---|
| 1/16 | 1.191 e-001 | 3.18 e-002 | 8.1 e-003 | 2.0 e-003 |
| 1/32 | 2.88 e-002 | 7.7 e-003 | 2.0 e-003 | 4.950 e-004 |
| 1/64 | 1.47 e-002 | 4.0 e-003 | 1.0 e-003 | 2.595 e-004 |
| 1/128 | 7.2 e-003 | 1.9 e-003 | 4.929 e-004 | 1.239 e-004 |

$$u(x) = (1-x)^4 x^8 \sin(\varepsilon x).$$

The observed maximum errors associated with $u_i s$ for Example 1, corresponding to different values of $\varepsilon$ are tabulated in Table 1. The absolute errors determined, using method developed by Ghazala and Nadia in [17] are shown in Table 2, which shows that the method presented in this paper is better than Ghazala and Nadia [17]. It is also confirmed from the Table 1 that if $h$ is reduced by factor 1/2, then $\|E\|$ is reduced by a factor 1/16, which indicates that the present method gives fourth order results.

**Example 2:** For $x \in [-1, 1]$, consider the following boundary value problem:

$$-\varepsilon u^{(4)}(x) + pu(x) = \varepsilon x((x-1)^4 x^4 - 24\varepsilon(5 - 60x + 210x^2 - 280x^3 + 126x^4)),$$

$$\text{with} \quad u(-1) = -16\varepsilon, u(1) = 0, u^{(2)}(-1) = -688\varepsilon, u^{(2)}(1) = 0.$$

The exact solution of the above system is,

$$u(x) = \varepsilon x^5 (1-x)^4.$$

The observed maximum errors associated with $u_i s$ for Example 2, corresponding to different values of $\varepsilon$ are tabulated in Table 3. The absolute errors determined, using method developed by Ghazala and Nadia in [17] are shown in Table 4, which shows that the method presented in this paper is better than Ghazala and Nadia [17]. It is also confirmed from the Table 3 that if $h$ is reduced by factor 1/2, then $\|E\|$ is reduced by a factor 1/16, which indicates that the present method gives fourth order results.

## CONCLUSION

In this paper fourth order singularly perturbed boundary value problem is solved using septic spline, which is computationally effective. The method has been examined for convergence and proved that the order of convergence is $O(h^4)$. Two examples are presented which support the order of convergence. Comparison with the existing method shows that the present method is better.

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