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The Dynamics of Some Extreme Doubly Stochastic Quadratic Operators

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Abstract: We provide an example of Lyapunov for doubly stochastic operators on a finite dimensional simplex. Besides we study the limit behavior of the trajectories of some extreme doubly stochastic operators on two dimensional simplex. We prove that except for certain points the trajectories of extreme doubly stochastic operators tend to the center of two dimensional simplex.

Key words: Majorization . Lyapunov function . doubly stochastic operator . extreme point . periodic point

INTRODUCTION

Quadratic stochastic operators (q.s.o. shortly) defined on finite dimensional simplex are considered to have many applications in population genetics [5]. From this point of view, the main problem in the theory of quadratic stochastic operators is to study the dynamics of these operators, which is extremely difficult. However, we can mention some results in this direction. The class of q.s.o. called Volterra has been outlined in [2], which is the discrete analogue of nonlinear Lotka-Volterra equation. The class of dissipative q.s.o. [8] was defined as $x \prec Vx$ where, $\prec$ is the notation of classical majorization [1, 6], that is the comparison of partial sums after non-increasing rearrangement (see notations below). It was shown [9] that for nontrivial dissipative q.s.o.'s the trajectory of any initial point either approaches the vertex of the simplex or approaches the face of the simplex. Using the majorization terminology, the class of doubly stochastic operator was defined in [3]. Further properties of doubly stochastic operators have been investigated in [4, 7]. It was shown that, up to permutation of components of the operators, there are 37 extreme points of the set of doubly stochastic quadratic operator on two dimensional simplex. The aim of the present note is to study the dynamics of some extreme doubly stochastic quadratic operators on two dimensional simplex. To do so, we give some preliminaries in the next section. The main results are given in the third section.

PRELIMINARIES

In this section we give some definition from majorization theory and give the definition of doubly stochastic operator.
We define the (m-1)-dimensional simplex as follows.

$$S^{m-1} = \{x = (x_1, x_2, \cdots, x_m) \in R^m : x_i \geq 0, \forall i = \overline{1, m}, \quad \sum_{i=1}^{m} x_i = 1\}.$$

The set $int S^{m-1} = \{x \in S^{m-1} : x_i > 0\}$ is called the interior of the simplex. The points $e_k = (0, 0, \cdots, \underset{k}{1}, \cdots, 0)$ are the vertices of the simplex and the scalar vector $(\frac{1}{m}, \frac{1}{m}, \cdots, \frac{1}{m})$ is the center of the simplex.
A quadratic stochastic operator $V: S^{m-1} \rightarrow S^{m-1}$ is defined as:

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$$(Vx)_k = \sum_{i,j=1}^{m} p_{ij,k}\, x_i x_j,$$

where the coefficients $p_{ij,k}$ satisfy the following conditions

$$p_{ij,k} = p_{ji,k} > 0, \qquad \sum_{k=1}^{m} p_{ij,k} = 1.$$

For any $x = (x_1, x_2 \cdots x_m) \in S^{m-1}$, we define $x_\downarrow = (x_{[1]}, x_{[2]}, \cdots x_{[m]})$, where $x_{[1]} > x_{[2]} > \cdots > x_{[m]}$-nonincreasing rearrangement of x. Recall that for two elements x,y of the simplex S^{m-1} the element x is majorized by y and write $x \prec y$ (or $y \prec x$) if the following holds

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]},$$

for any $k = \overline{1, m-1}$. In fact, this definition is referred as weak majorization [6], the definition of majorization requires $\sum_{i=1}^{m} x_{[i]} = \sum_{i=1}^{m} y_{[i]}$. However, since we consider points only from the simplex, we may drop this condition.
A matrix $P = (p_{ij})_{i,j=\overline{1,m}}$ is called doubly stochastic(sometimes bistochastic), if

$$p_{ij} \ge 0, \qquad \forall i, j = \overline{1, m},$$

$$\sum_{i=1}^{m} p_{ij} = 1, \forall j = \overline{1, m}, \qquad \sum_{j=1}^{m} p_{ij} = 1, \forall i = \overline{1, m}.$$

For a doubly stochastic matrix P = (p_{ij}) if its entries consist of only 0's and 1's, then the matrix is a permutation matrix.

A linear map T: $S^{m-1} \to S^{m-1}$ is said to be T-transform, if $T = \lambda I + (1-\lambda)P$, where l is an identity matrix, P is a permutation matrix which is obtained by swapping only two rows of l and $0 \le \lambda \le 1$.

Lemma 1: [6] For the concept of majorization and $x,y \in S^{m-1}$, the following assertions are equivalent.

1) $x \prec y$ that is $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$, $k = \overline{1, m-1}$.
2) x = Py for some doubly stochastic matrix P.
3) The vector x belongs to the convex hull of all m! permutation vectors of y.
4) The vector x can be obtained by a finite compositions of T-transforms of the vector y, that is, there exist T-transforms $T_1, T_2, \ldots, T_k$ such that $x = T_1 T_2 \ldots T_k y$.

So from the above lemma it follows that doubly stochasticity of a matrix P is equivalent to $Px \prec x$ for all $x \in S^{m-1}$. Motivated by this, in [3], the definition of doubly stochastic operator was given as follows.

A continuous stochastic operator V: $S^{m-1} \to S^{m-1}$ is called doubly stochastic, if $Vx \prec x$ for all $x \in S^{m-1}$. Identity operator, permutation operators (that is the linear operators with permutation matrix) and T-transforms are all doubly stochastic. The operator V: $S^2 \to S^2$ given by

$$Vx = x^2 + 2yz,$$
$$Vy = y^2 + 2xz,$$
$$Vz = z^2 + 2xy.$$

is an example for doubly stochastic quadratic operator. Doubly stochasticity can be shown using the definition or can be obtained as the consequence of Theorem 2.6 of [2].

Let V be doubly stochastic operator and $x^0 \in S^{m-1}$. The sequence $\{x^0, V(x^0), V^2(x^0), \dots\}$ is called the trajectory starting at x^0, Usually, we put $V^0(x^0) = x^0$. We denote by $\omega(x^0)$ the set of limit points of the trajectory and it is said to be the ω-limit set of the trajectory.

MAIN RESULT

A continuous functional φ is said to be a Lyapunov function for the operator V, if the limit $\lim_{n\to\infty} \varphi(V^n(x^0))$ exists along the trajectory $\{x^0, V(x^0), V^2(x^0), \dots\}$. The Lyapunov function is considered to be very useful in the study of limit behavior of discrete dynamical systems [2].

Theorem 1: A functional given by

$$\varphi(x) = \sum_{i=1}^{m} x_i^2$$

is a Lyapunov function for doubly stochastic operator V.

Proof: Let V be doubly stochastic. According to Lemma 2.1 the condition $Vx \prec x$ implies the existence of doubly stochastic matrix $P(x) = \{p_{ij}(x)\}$ (different matrix P for different x.) such that $Vx = P(x)x$. Obviously, the matrix P(x) satisfies the following conditions

$$p_{ij}(x) \geq 0, \qquad \forall i, j = 1, m$$

$$\sum_{i=1}^{m} p_{ij}(x) = 1, \forall j = \overline{1, m}, \quad \sum_{j=1}^{m} p_{ij}(x) = 1, \forall i = \overline{1, m}.$$

Let $Vx = P(x)x$, then $(Vx)_i = \sum_{j=1}^{m} p_{ij}(x) x_j$. Denote

$$L = \sum_{i=1}^{m} (Vx)_i^2.$$

Then we have

$$L = \sum_{i=1}^{m} [\sum_{j=1}^{m} p_{ij}(x) x_j]^2 == \sum_{i=1}^{m} [\sum_{j=1}^{m} p_{ij}^2(x) x_j^2 + 2 \sum_{k<j} p_{ik}(x) p_{ij}(x) x_k x_j].$$

Further, applying $2x_k x_j \leq x_k^2 + x_j^2$ for $k, j = \overline{1, m}$ we get

$$L \leq \sum_{i=1}^{m} [\sum_{j=1}^{m} p_{ij}^2(x) x_j^2 + \sum_{k<j} p_{ik}(x) p_{ij}(x) x_k^2 + \sum_{k<j} p_{ik}(x) p_{ij}(x) x_j^2] =$$
$$= \sum_{i=1}^{m} [x_1^2 (p_{i1}^2(x) + p_{i1}(x) p_{i2}(x) + \dots + p_{i1}(x) p_{im}(x)) +$$
$$+ x_2^2 (p_{i2}^2(x) + p_{i2}(x) p_{i3}(x) + \dots + p_{i2}(x) p_{im}(x) + p_{i1}(x) p_{i2}(x)) + \dots +$$
$$+ x_m^2 (p_{im}^2(x) + p_{i1}(x) p_{im}(x) + \dots + p_{im-1}(x) p_{im}(x))].$$

Note that

$$p_{ik}^2(x) + \sum_{l=\overline{1,m}, l\neq k} p_{ik}(x) p_{il}(x) == p_{ik}(x) \sum_{l=1}^{m} p_{il}(x) = p_{ik}(x).$$

Hence

$$L < \sum_{i=1}^{m} [x_1^2 p_{i1}(x) + x_2^2 p_{i2}(x) + \dots + x_m^2 p_{im}(x)] = \sum_{i=1}^{m} \sum_{j=1}^{m} x_j^2 p_{ij}(x) = \sum_{j=1}^{m} x_j^2 \sum_{i=1}^{m} p_{ij}(x) = \sum_{j=1}^{m} x_j^2.$$

Therefore

$$\sum_{i=1}^{m} (Vx)_i^2 < \sum_{i=1}^{m} x_i^2.$$

The last inequality means that the functions $\varphi(x) = \sum_{i=1}^{m} x_i^2$ is monotonically decreasing along the trajectory. Since the function φ(x) is bounded on the simplex, then the limit $\lim_{n\to\infty} \varphi(V^n(x^0))$ exists. Hence φ is a Lyapunov function for doubly stochastic operator V.

Using this theorem, owe can investigate the limit behavior of the trajectories of doubly stochastic operators on s finite dimensional simplex. This will be our next investigation. Further, we are going to study the dynamics of some extreme doubly stochastic operators.

The point x^0 is called p-periodic, if there is a positive integer p such that $V^p(x^0) = x^0$ and $V^i(x^0) \neq x^0 \ \forall i = \overline{1, p-1}$. If p = 1, we say that the point is fixed. We just say periodic if the period is irrelevant.
Let V_1: $S^2 \to S^2$ be given by

$$\begin{aligned} V_1 x &= x^2 + 2yz, \\ V_1 y &= y^2 + 2xz, \\ V_1 z &= z^2 + 2xy. \end{aligned}$$

The operator V_1 is doubly stochastic [4]. Even more, it is extreme point of the set of doubly stochastic operator.

Theorem 2: If x^0 is not the vertex of the simplex, then under the operator V_1 the trajectory of the point x^0 tends to the center of the simplex, that is to the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Proof: Let us first find the fixed points of V_1. By solving the equation

$$\begin{aligned} x &= x^2 + 2yz, \\ y &= y^2 + 2xz, \\ z &= z^2 + 2xy, \end{aligned}$$

one can find the following solutions

$$(1,0,0), (0,1,0), (0,0,1), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

If we define the function φ: $S^2 \to R$ as $\varphi(x) = \varphi(x, y, z) = x^2 + y^2 + z^2$, then one can see that

$$2\varphi(Vx) = 3\varphi(x)^2 - 2\varphi(x) + 1 == (3\varphi(x) - 1)(\varphi(x) - 1) + 2\varphi(x).$$

Moreover

$$\frac{(x+y+z)^2}{3} \leq x^2 + y^2 + z^2 \leq (x+y+z)^2 = 1$$

implies that the maximum value of φ(x) is 1 at fixed points (1,0,0),(0,1,0),(0,0,1) and minimum is $\frac{1}{3}$ at the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. If $\varphi(x^0) \neq 1, \frac{1}{3}$ then

$$2\varphi(V(x^0)) = (3\varphi(x^0) - 1)(\varphi(x^0) - 1) + 2\varphi(x^0) < 2\varphi(x^0).$$

This means that the trajectory of any initial point, except for fixed points (1,0,0),(0,1,0),(0,0,1) converges to the center of the simplex.

Note that by permuting the component of the (extreme) doubly stochastic operator, one can define another example of (extreme) doubly stochastic operator, investigation of which goes along with the same lines as in Theorem 2. However, in this setting, the vertices of the simplex may turn into periodic.

Theorem 3: Let doubly stochastic operator V_π: $S^{m-1} \to S^{m-1}$ be given as

$$\begin{aligned} V_{\pi(1)} x &= x^2 + 2yz, \\ V_{\pi(2)} y &= y^2 + 2xz, \\ V_{\pi(3)} z &= z^2 + 2xy. \end{aligned}$$

where π is the permutation of the numbers (1,2,3). Then the trajectory of any initial point, except the vertices of the simplex, tends to the center of the simplex.
We consider another type of operator. Let V_2: $S^2 \to S^2$ be given by

$$V_2 x = z^2 + xy + xz,$$
$$V_2 y = y^2 + xy + yz,$$
$$V_2 z = x^2 + xz + yz.$$

This operator is doubly stochastic [4].

Theorem 4: If an initial point x^0 is not periodic, then the trajectory of this point under V_2 tends to the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Proof: The set of fixed points of V_2 is $\{(x, y, z) \in S^2 : x = z\}$, so it has infinitely many fixed points. One can also see that all the elements of the set $\{(x, y, z) \in S^2 : y = 0\}$ are 2-periodic. Simple calculations show that the inequality

$$(Vx)^2 + (Vy)^2 + (Vz)^2 \le x^2 + y^2 + z^2$$

can be reduced to

$$-2y(x - z)^2(x + z) \le 0.$$

The equality holds when y = 0 which implies that the point is periodic, or when x = z, that is when the point is fixed. In all other cases, the inequality is strict. This brings to the conclusion that except for periodic points the trajectory tends to the center of the simplex.

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