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Multipliers on Fréchet Algebra

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Abstract: This paper is devoted to establish some fundamentally important results on a commutative semi simple Fréchet algebra. It has been shown that a multiplier on a semi simple Fréchet algebra is a product of an idempotent and an invertible multiplier. It has also been shown that a multiplier on a commutative semi simple Fréchet algebra which is also a Fredholm operator is a product of an idempotent and an invertible element of a continuous linear self mapping of commutative semi simple Fréchet algebra. Finally have shown that for a multiplier T on a commutative semi simple Fréchet algebra A, $T^2(A)$ is closed $\Leftrightarrow$ $T(A)\oplus\ker(T)$ is closed, $T(A)\oplus\ker(T)$ is closed $\Leftrightarrow A = T(A)\oplus\ker(T)$ and T is a product of an idempotent and an invertible multiplier $\Leftrightarrow A = T(A)\oplus\ker(T)$.

Key words: Commutative semi simple Fréchet algebra · Banach algebras · topological algebra · locally convex algebra

INTRODUCTION

There has been a rapid growth of interest in Banach algebras. Consequently, the subject has brought the theory to a point where it is no longer just a promising tool in its own right. Banach algebra has developed due to analytic and algebraic influences. The analytic emphasis has been on the study of some special Banach algebras, along with some generalization and on extension of function theory and harmonic analysis to a more general situation. The algebraic emphasis has been on various aspects of structure theory.

The concept of multipliers on Banach algebras was firstly introduced by S. Helgason [6]. The general theory of multipliers have been studied quite extensively in the context of Banach algebra without order by F.T. Birtel [4], J. K. Wang [17], R. Larsen [10] etc. T. Husain [8] has generalized the notion of a multiplier on a topological algebra without order. N. Mohammad [13, 14] has studied the spectral properties of multipliers and compact multipliers on topological algebras. L. A. Khan [9] has studied the double multipliers on topological algebras. Recently H. Render [15] has introduced the study of multipliers in the framework of vector spaces of holomorphic functions.

In view of applications and recent developments in the theory of topological algebras, it is important to study multipliers on more generalized topological algebras. As an example, Quantum groups are the outcome of the study of multipliers on Hopf algebra. Multipliers have immediate applications in many areas of mathematics, such as, Harmonic analysis, Differential geometry, Representation theory, Field theory, optimal control, Quantum Mechanics Statistical Mechanics and consequently in engineering, etc. As an application of control theory, one may consider the areas such as scheduling and control of engineering devices, aerospace engineering, maximum orbit transfer problem, navigation, mobile robotics and automated vehicles,etc. Solid state engineering requires the concepts of quantum mechanics. Distribution theory, propogation of heat, robotics, image analysis and decomposition of electromagnetic waves, etc require the concept of harmonic analysis. Differential geometry by means of tensor calculus is an application in engineering. It is worth observing that not much work has been done in the setting of topological algebra.

This paper is devoted to establish some fundamentally important results on a commutative semi simple Fréchet algebra. It has been shown that a multiplier on a semi simple Fréchet algebra is a product of an idempotent and an invertible multiplier. It has also been shown that a multiplier on a commutative semi simple Fréchet algebra which is also a Fredholm operator is a product of an idempotent and an invertible element of a continuous linear self mapping

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of commutative semi simple Fréchet algebra. Finally have shown that for a multiplier T on a commutative semi simple Fréchet algebra A,

$T^2(A)$ is closed $\Leftrightarrow$ $T(A)\oplus\ker(T)$ is closed, $T(A)\oplus\ker(T)$ is closed $\Leftrightarrow A = T(A)\oplus\ker(T)$ and T is a product of an idempotent and an invertible multiplier $\Leftrightarrow A = T(A)\oplus\ker(T)$.

The concept of topological algebra is a natural generalization of Banach algebra. David van Dantzig [5] has used the word "Topological Algebra' very first time in his Ph.D. thesis and later in a whole series of papers. Later on it also appeared in the literature by R. Arens [1]. Fundamental results, although at the same time but separately, were published by R. Arens [2] and E.A. Michael [11]. The study of these algebras helped to investigate the non-normable behaviors in mathematics and physics.

MATERIAL AND METHODS

A vector space A over the complex field C is called algebra if

i. A is closed i.e., $\forall x,y\in A,\ xy\in A$
ii. $x(yz)=(xy)z=xyz\ \ \forall x,y,z\in A$
iii. $x(y+z)=xy+xz$ and $(x+y)z=xz+yz\ \ \forall x,y,z\in A$
iv. $(\lambda x)(\mu y)=(\lambda\mu)(xy)\ \ \forall x,y\in A$ and $\lambda,\mu\in C$

Algebra A with Hausdorff topology and continuous algebraic operations will be called topological algebra. A topological algebra will be called locally convex algebra if its topology is generated by the semi norms $\{p_\alpha\}_{\alpha\in I}$ such that $\forall\alpha\in I$

i. $p_\alpha(x)\geq 0$ and $p_\alpha(x)=0$ if $x=0$
ii. $p_\alpha(x+y)\leq p_\alpha(x)+p_\alpha(y)\ \ \forall x,y\in A$
iii. $p_\alpha(\lambda x)=|\lambda|\,p_\alpha(x)\ \ \forall x\in A$ and $\lambda\in C$

A locally convex algebra will be called locally multiplicatively convex if $p_\alpha(xy)\leq p_\alpha(x)p_\alpha(y)\ \ \forall\alpha\in I\ \&\ \forall x,y\in A$. A complete metrizable locally multiplicatively convex algebra is called Fréchet algebra. ΔA will be the set of all non-zero continuous multiplicative linear functionals on A. If A is a commutative locally multiplicatively convex algebra then; $\mathrm{rad}(A)=\{x\in A: f(x)=0\ \forall f\in\Delta A\}$. In the case of $\mathrm{rad}(A)=\{0\}$, it will be called semi simple. E_A will be the set of all minimal idempotents of A and soc(A) will be the sum of minimal ideals of A. A multiplier on A will be a mapping $T:A\to A\ \ni Tx.y=x.Ty\ \ \forall x,y\in A$ and the set of all multipliers on A will be denoted by M(A). Multipliers on a semi simple algebra are linear. Multipliers on a proper (without order) complete metrizable algebra [16] and consequently on Fréchet algebra are continuous. If A is Fréchet algebra then the range of T denoted as T(A) and ker (T) are two sided ideals of A. If A be a commutative semi simple Fréchet algebra and T is a linear bounded linear operator on A, we define $\alpha(T)=\dim\ker(T)$ and $\beta(T)=\dim(A/T(A))$. If $\alpha(T)<\infty$ and $\beta(T)<\infty$, then T is said to have a finite deficiency. Collection of all continuous linear self mapping of A will be denoted by B(A). If A is commutative Fréchet algebra and Te B(A) then T is called Fredholm operator if T has finite deficiency. The set of all Fredholm oprators on A will be denoted by F (A). If A is a commutative locally multiplicatively convex algebra then to each $x\in A$, we define Gelfand transform of x as $\hat{x}:\Delta(A)\to C$ given by $\hat{x}(f)=f(x)\ \forall f\in\Delta A$. The set of all Gelfand transforms of A is denoted by $\hat{A}$ where $\hat{A}=\{\hat{x}:x\in A\}$. If T is a linear operator on an algebra A then it ascent denoted as p(T) is defined (if exist) to be the smallest integer p≥0 for which $\ker(T^p)=\ker(T^{p+1})$. Similarly the descent of T denoted as q (T) is defined (if exist) to be smallest integer q≥0 for which $T^q(A)=T^{q+1}(A)$

RESULTS AND DISCUSSION

Theorem 1: Let A be a semi simple Fréchet algebra. Let $T,S\in M(A)$ such that $\hat{S}(f)=\hat{T}(f)^{-1}$ on $\Delta(A)/\hat{T}^{-1}(\{0\})$. Then T is a product of an idempotent $P\in M(A)$ and an invertible multiplier.

Proof:

$$(TS\hat{T})(f) = \hat{T}(f) \ \forall f \in \Delta(A) \ \text{ because if } f \in \Delta(A)/\hat{T}^{-1}(\{0\})$$

then

$$(TS\hat{T})(f) = \hat{T}(f).\hat{S}(f).\hat{T}(f) = \hat{T}(f)$$

and if

$$f \in \hat{T}^{-1}(\{0\}), \text{i.e.} \, \hat{T}(f) = 0$$

then

$$(TS\hat{T})(f) = \hat{T}(f).\hat{S}(f).\hat{T}(f) = 0 = \hat{T}(f)$$
$$\text{,i.e.}(TS\hat{T})(f) = \hat{T}(f) \, \forall f \in \Delta(A), \text{i.e.}$$
$$(TST - \hat{T})(f) = 0 \, \forall f \in \Delta(A) \Rightarrow TST - T \in \text{rad}(M(A))$$

Since M(A) is semisimple, hence, TST=T, i.e. T is relatively regular operator of M(A). Since $(TS)^2$ = TSTS = TS, hence P = ST+TS is an idempotent element of M(A).

Now let U = T + (I-P) and V = STS + (I-P), where P = ST = TS. Then VU = [STS + (I-P)][T+(I-P)] = STST + (I-P)T + STS (I-P) + $(I-P)^2$ = ST + T –TST + STS – STSTS + $(I-P)^2$ = P^2 + I - 2P + P^2 = I. Hence, U is invertible with U^{-1}=V. Also UP = [T + (I-P)] P = TST + P – P = T

Theorem 2: Let A be a commutative semi simple Fréchet algebra. If $T \in \phi(A)M(A)$ has index zero then there exist an idempotent P and an invertible element U$\in$B(A) such that T = PU.

Proof: If $T \in \phi(A)M(A)$ has index zero then $\alpha(T) = \beta(T)$. $p(T) \leq 1 \, \forall \, T \in M(A)$. Let

$$x \in \ker(T^2) \Rightarrow T^2x = 0 \Rightarrow 0 = x\,T^2x = T^2x^2 = (Tx)^2 \Rightarrow Tx = 0 \Rightarrow x \in \ker(T)$$

Hence $\ker(T^2) \subseteq \ker(T) \subseteq \ker(T^2)$. By proposition 38.6, page 163 of H. Heuser [7] $p(T) = q(T) \leq 1$. If p(T) = q(T) = 0 then T is 1-1 and onto so invertible.
Hence T = I T.

If p(T) = q(T) = 1 then by proposition 38.4, page 162 of H. Heuser [Heuser, 1982] A = T(A)$\oplus$ker(T). Let T_1 = T/T(A). We will show that T_1 is invertible. Let y_1, y_2 e T(A) ∍ $T_1(y_1) = T_1(y_2)$ then $T_1(y_1\text{-}y_2) = 0$ implies y_1-y_2 e ker $(T_1)\subseteq$ker(T) but y_1-y_2 e ker (T)$\cap$ T(A) = {0} implies $y_1 = y_2$ implies T_1 is 1-1. Since q(T) =1 so $T_1(T(A))$= T(T(A)) = T^2(A) = T(A) implies T_1 is onto. Hence, T_1 is invertible. Since T_1 is linear and bounded so is its inverse S on T(A).
Now, if y e T(A) then $\exists$ z e T(A)

$$\ni y = T_1(z) \Rightarrow z = T_1^{-1}(y) = S(y)$$

For each

$$f \in \Delta(A), (S\hat{y})(f) = \hat{z}(f)$$

where

$$\hat{z}(f) \neq 0 \ \forall f \in \Delta(A)$$

This along with theorem 2.5 of T. Husain [8] implies

$$(T\hat{z})(f) = \mu^T(f).\hat{z}(f)$$

we have

$$(S\hat{y})(f).\mu^T(f) = (T\hat{z})(f).$$

Let

$$K = \hat{T}^{-1}(\{0\}) = \left\{ f \in \Delta)A) : \hat{T}(f) = 0 \right\}.$$

By theorem 1.41 of R. Larsen [10]

$$(Tz\hat{)}(f) = \hat{T}(f).\hat{z}(f) = 0$$

Since

$$\mu^T(f) \neq 0 \ \forall f \in \Delta(A) \Rightarrow (Sy\hat{)}(f) = 0$$

Now if

$$f \in K^c = \Delta(A)/\hat{T}^{-1}(\{0\}) \text{i.e.} \hat{T}(f) \neq 0$$

then

$$(Sy\hat{)}(f) = \hat{z}(f) = (Tz\hat{)}(f)/\hat{T}(f) = [\hat{T}(f)]^{-1}.\hat{y}(f) \ (\because y = T_1 z = Tz \text{ for } z \in T(A)$$

Hence,

$$(Sy\hat{)}(f) = \begin{cases} 0 & \text{if } f \in K \\ [\hat{T}(f)]^{-1}.\hat{y}(f) & \text{if } f \in K^c \end{cases} \tag{1}$$

Now, consider the bounded operator $T_2 = S^2T$. If $f \in K$ then because of (1) and the fact that $S(Tx) \in T(A)$ we have

$$(T_2x\hat{)}(f) = (S^2Tx\hat{)}(f) = (S(STx)\hat{)}(f) = 0$$

If $f \in K^C$ then because of (1) and by theorem 1.41 of R. Larsen [Larsen, 1971] we have

$$(T_2x\hat{)}(f) = (S(STx)\hat{)}(f) = [\hat{T}(f)]^{-1}(STx\hat{)}(f) = [\hat{T}(f)]^{-2}(Tx\hat{)}(f) = [\hat{T}(f)]^{-2}.\hat{T}(f).\hat{x}(f) \Rightarrow (T_2x\hat{)}(f) = [\hat{T}(f)]^{-1}.\hat{x}(f)$$

Now we claim that $T_2 \in M(A)$. For each $f \in \Delta(A)$, let $x \in \ker(f)$ then

$$(T_2x\hat{)}(f) = (S^2Tx\hat{)}(f) = 0$$

if $f \in K$ and $(T_2x\hat{)}(f) = [\hat{T}(f)]^{-1}.\hat{x}(f) = 0$ if $f \in K^C$

In both cases we have

$$(T_2x\hat{)}(f) = 0 \Rightarrow f(T_2x) = 0 \ \forall f \in \Delta(A) \Rightarrow T_2x \in \ker(f) \ \forall f \in \Delta(A) \Rightarrow T_2(\ker(f)) \subseteq \ker(f) \ \forall f \in \Delta(A)$$

Hence, by theorem 2.18 of Nasrullah Aziz [12] $T_2 \in M(A)$. T and T_2 satisfy the hypothesis of theorem 1, consequently, T is a product of an idempotent and an invertible multiplier.

Theorem 3: Let A be a commutative semi simple Fréchet algebra. If $T \in M(A)$ then $T^2(A)$ is closed $\Leftrightarrow$ $T(A) \oplus \ker(T)$ is closed.

Proof: Let $T^2A)$ be closed. Let $a \in \overline{T(A) + \ker(T)} \Rightarrow$ there exist a sequence

$$< Ta_n + b_n > \text{in} \, T(A) + \ker(T)$$

that is a sequence $< a_n > \in A$ and a sequence $< b_n > \in \ker(T) \ni T a_n + b_n \to a$ in $\overline{T(A) + \ker(T)}$. Continuity of $T \Rightarrow T(Ta_n + b_n) \to Ta$

Since

$$T(b_n) \to 0 \therefore T^2(a_n) \to T(a)$$

in $\overline{T^2(A)}$ but $T^2(A)$ is closed, $\therefore T^2(a_n) \to T(a)$ in $T^2(A) \Rightarrow$

$$\exists \text{ some } b \in A \ni T^2(b) = T(a) \Rightarrow T(a - Tb) = 0 \Rightarrow a - Tb \in \ker T \Rightarrow a \in T(A) + \ker T$$

Hence

$$\overline{T(A) + \ker(T)} \subseteq T(A) + \ker(T) \Rightarrow T(A) + \ker(T)$$

is closed. Since

$$T(A) \cap \ker(T) = \{0\} \text{ (Theorem1)} \Rightarrow T(A) \oplus \ker(T)$$

is closed

Conversely, Let T(A)⊕ker(T) be closed ⇒ T(A)⊕ker(T) is closed. Let

$$Ta \in \overline{T^2(A)} \Rightarrow \exists \text{a sequence } < T^2 a_n > \text{in} T^2(A) \ni T^2 a_n \to Ta \Rightarrow T(Ta_n - a) \to 0$$

Since T:A→T(A) is continuous and onto, hence, by corollary 2.12 page 48 of W. Rudin [16] T is open. Since T:A→T(A) is open ⇒

$$\exists \text{ asequence} < b_n > \text{inAforwhich} Tb_n = T(Ta_n - a)$$

Since

$$T(Ta_n - a) \to 0 \Rightarrow Tb_n \to 0 \Rightarrow b_n \to 0$$

Hence, $T(Ta_n - b_n - a) = 0$

$$\therefore Ta_n - b_n - a \in \ker T \Rightarrow a + b_n \in T(A) + \ker T$$

As n→∞

$$\lim_{n\to\infty}(a + b_n) \in \overline{T(A) + \ker(T)} = T(A) + \ker(T) \text{ (Byassumption)} \Rightarrow$$

$$a \in T(A) + \ker T \ (\because b_n \to 0) \Rightarrow Ta \in T^2(A). \text{Hence} \overline{T^2(A)} \subseteq T^2(A) \Rightarrow T^2(A)$$

is closed.

Theorem 4: If A be a commutative semi simple Fréchet algebra with $T \in M(A)$ and $\overline{soc(A)} = A$ then T(A)⊕ker(T) is closed ⇔ A = T(A)⊕ker(T)

Proof: Let T(A)⊕ker(T) be closed. By theorem 1 of M. Azram [3]

$$soc(A) \subseteq T(A) \oplus \ker(T) \therefore A = \overline{soc(A)} \subseteq \overline{T(A) \oplus \ker T} = T(A) \oplus \ker T.$$

Hence, A = T(A)⊕ker(T). Conversely.

Let A = T(A)⊕ker(T).

Since

$$A = \overline{soc(A)} \subseteq \overline{T(A) \oplus \ker T} \Rightarrow T(A) \oplus \ker T = A = \overline{T(A) \oplus \ker T} \Rightarrow T(A) \oplus \ker(T)$$

is closed.

Theorem 5: If A be a commutative semi simple Fréchet algebra with $T \in M(A)$ and $\overline{soc(A)} = A$ then T is a product of an idempotent and an invertible multiplier ⇔ A = T(A)⊕ker(T)

Proof: Let T = PU where P e M(A) is an idempotent and U e M(A) is invertible. Hence by theorem 2.4 of N. Mohammad [13] A = T(A)⊕ker(T).

Conversely.

Let A = T(A)⊕ker(T). By Prop 38.4 of H. Heuser [Heuser, 1982] p(T) = q(T)≤1 and by a theorem of N. Mohammad [13], $\alpha(T) = \beta(T) \Rightarrow T \in \phi(A)$. By theorem 2, T = PU where P is an idempotent and U is an invertible multiplier.

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