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Weight Martingale-Ergodic and Ergodic-Martingale Theorems

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Abstract: In this paper we prove weighted martingale-ergodic and weighted ergodic-martingale theorems. Furthermore, analogous dominant and maximal inequalities for weighted martingale ergodic sequences and weighted ergodic martingale averages are also obtained.

Key words: Ergodic averages . martingale . ergodic-martingale . martingale-ergodic averages

INTRODUCTION

General theories unifying ergodic averages and martingales were reported by Kachurovskii [1-3]. Four different variants for theories unifying ergodic averages and martingales have been reported in [4-7]. Besides, one parameter weighted ergodic theorem and multiparameter weighted ergodic theorems have been investigated by Baxter J.H. Olsen [8] and R.L. Jones, J.H. Olsen [9], respectively. In [10], M. Lin and M. Weber considered weighted ergodic theorems and strong laws of large numbers. General ergodic theory is reported in [11].

In this paper we prove weighted martingale-ergodic and weighted ergodic-martingale theorems. Furthermore, analogous dominant and maximal inequalities for weighted martingale ergodic sequences and weighted ergodic martingale averages are also obtained.

Preliminaries: Let (Ω,Σ,λ) be a space with a finite measure, $L_0 = L_0(\Omega)$ be a space of complex measurable functions on Ω,

$$L_p = \{f \in L_0 : \int_\Omega |f|^p d\lambda < \infty\},$$

$p \geq 1$, with the norm

$$\| f \|_p = \left(\int_\Omega |f|^p d\lambda\right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$, $\| f \|_\infty = \sup\{|f(\omega)| : \omega \in \Omega\}$ if $p = \infty$.

Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a monotone sequence of σ-subalgebras of Σ, $\mathcal{A}_n \uparrow \mathcal{A}_\infty$ (or $\mathcal{A}_n \downarrow \mathcal{A}_\infty$) $E: L_p \to L_p$ be the expectation operator, $T: L_p \to L_p$ be the Dunford-Schwartz operator. Put

$$S_n(f,T) - \frac{1}{n}\sum_{k=1}^{n} T^k f, f^* - \lim_{n\to\infty} S_n(f,T), f_\infty^* = E(f^*|\mathcal{A}_\infty)$$

In [1] it is proved the following

Theorem 1.1

1) Let $f \in L_p$, $p \in [1,\infty)$. Then $E(S_n(f,T)|\mathcal{A}_n) \to f_\infty^*$ in L_p, in this case $\| f_\infty^* \|_p < \| f \|_p$.
2) Let $\in L_p$, $p \in (1,\infty)$. Then

$$E(S_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^* \text{ in } L_p.$$

3) Let $f \in L_1$, $\sup_{n\geq 1}|S_n(f,T)| \in L_1$. Then $E(S_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_0.

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Put $f_\infty = E(f|\mathcal{A}_\infty)$, $S_n(f_\infty, T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k f_\infty$, $f_\infty^* = \lim_{n\to\infty} S_n(f_\infty, T)$.

Theorem 1.2: [2]

1) Let $f\in L_p$, $p\in[1,\infty)$. Then $S_n(E(f|\mathcal{A}_n), T) \to f_\infty^*$ in L_p as $n\to\infty$.
2) Let $f\in L_p$, $p\in(1,\infty)$. Then

$$S_n(E(f|\mathcal{A}_n), T) \xrightarrow{(o)} f_\infty^* \text{ in } L_p.$$

3) Let $\in L_1$, $\sup |E(f|\mathcal{A}_n)| \in L_1$. Then $S_n(E(f|\mathcal{A}_n), T) \xrightarrow{(o)} f_\infty^*$ in L_0.

Lemma 1: [2, 12] Let $\{f_n\}\in L_p$, $p\in[1,\infty)$ and $f_n \to f^*$ in L_p, as $n\to\infty$. Then $E(f_n|\mathcal{A}_n) \to E(f^*|\mathcal{A}_\infty)$ in L_p as $n\to\infty$.

Lemma 2: [2, 12] Let $f_n \xrightarrow{(o)} f^*$ as $n\to\infty$ and $\sup_{n\geq 1} |f_n| \in L_1$. Then

$$E(f_n|\mathcal{A}_n) \xrightarrow{(o)} E(f^*|\mathcal{A}_\infty) \text{ in } L_0$$

Lemma 3: [11] Let $f_n \xrightarrow{(o)} f^*$ as $n\to\infty$ and $\sup_{n\geq 1} |f_n| \in L_1$. Then $S_n(f_n, T) \xrightarrow{(o)} f^*$ in L_0 as $n\to\infty$ where $f^* = \lim_n S_n(f, T)$.

Definition 1.3: The sequence numbers $\alpha(k)$ is called Besicovich sequences such that given $\varepsilon>0$, there is a trigonometric polynomial $\psi_\varepsilon(k)$ such that $\lim_{n\to\infty} \sup_n \frac{1}{n}\sum_{k=0}^{n-1} |\alpha(k) - \psi_\varepsilon(k)| < \varepsilon$.

A sequence $\alpha(k)$ is called bounded Besicovich if $\{\alpha(k)\}\in l^m$. In [9] it was proved the following

Theorem 1.4: Let T be denoted the Dunford-Schwartz operator. Then there exists

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \alpha(k)T^k(f)$$

a.e. for every $f\in L_p$, $1<p<\infty$ and all bounded Besicovich sequences $\alpha(k)$. In this case, if $\in(1,\infty)$, then $A_n(f,T) = \frac{1}{n}\sum_{k=0}^{n-1} \alpha(k)T^k(f)$ (o) -converges in L_p and if p = 1, then $A_n(f, T)$ (o)-converges in L_0.

In this paper we prove theorems analogous to 1.1 and 1.2 in the case of weighted averages.

WEIGHTED MARTINGALE-ERGODIC THEOREMS

Let $\{\mathcal{A}_n\}_{n=1}^\infty$ be a monotone sequence of σ-subalgebras of Σ, $\mathcal{A}_n \uparrow \mathcal{A}_\infty$ (or $\mathcal{A}_n \downarrow \mathcal{A}_\infty$). $\alpha(k)$ be a Besicovich bounded sequence, $T: L_p\to L_p$ be the Dunford-Schwartz operator. We put,

$$A_n(f,T) = \frac{1}{n}\sum_{k=0}^{n-1} \alpha(k)T^k(f),\ f^* = \lim_{n\to\infty} A_n(f,T),\ f_\infty^* = E(f^*|\mathcal{A}_\infty).$$

Theorem 2.1

1) Let $f\in L_p$, $p\in[1,\infty)$. Then $E(A_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_p.
2) If $f\in L_1$ and $\sup_{n\geq 1} A_n(f,T) \in L_1$, then $E(A_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_0.

Proof

1) Since

$$\sup_{n\geq 1} |A_n(f)| = \sup_{n\geq 1} |\frac{1}{n}\sum_{k=0}^{n-1} \alpha(k)T^k(f,T)| \leq \sup_{n\geq 1} \frac{1}{n}\sum_{k=0}^{n-1} |\alpha(k)||T|^k(|f|) \leq_b \sup_{n\geq 1} \frac{1}{n}\sum_{k=1}^{n-1} |T|^k(f).$$

where $b = \sup |\alpha(k)|$, by the Akcoglu's theorem we have $\sup_{n\geq 1} |\mathcal{A}_n(f,T)| \in L_p$.

According to Theorem 1.2 [9], $A_n(f,T)\xrightarrow{(o)} f^*$. Therefore Lemma 2 implies $E(A_n(f,T)|\mathcal{A}_n)\xrightarrow{(o)} E(f^*|\mathcal{A}_\infty) = f_\infty^*$ in L_0 as $n\to\infty$.

Let $\sup_{n\geq 1}|A_n(f,T)| = h$. Then $E(A_n(f,T)|\mathcal{A}_n)\leq E(h|\mathcal{A}_n)$ for all n. By Theorem 2 [13] $\sup_{n\geq 1} E(h|\mathcal{A}_n)\in L_p$. Hence, $\sup_{n\geq 1}|E(A_n(f,T)|\mathcal{A}_n)|\in L_p$. Therefore $E(A_n(f,T)|\mathcal{A}_n)\xrightarrow{(o)} f_\infty^*$ in L_p.

2) By Theorem 1.4 [9] we have $A_n(f,T)\to f^*$ a.e. As $\sup_{n\geq 1} A_n(f,T)\in L_1$, by Lemma 2 we obtain $E(A_n(f,T)|\mathcal{A}_n)\to f_\infty^*$ a.e. Since the convergence a.e. is the (o)-convergence in L_0, we obtain $E(A_n(f,T)|\mathcal{A}_n)\xrightarrow{(o)} f_\infty^*$ in L_0.

WEIGHTED ERGODIC-MARTINGALE THEOREM

Let, as in Section 2, $\{\mathcal{A}_n\}_{n=1}^\infty$ be a monotone sequence of σ-subalgebras of Σ, $\mathcal{A}_n\uparrow\mathcal{A}_\infty$ (or $\mathcal{A}_n\downarrow\mathcal{A}_\infty$), $\alpha(k)$ be a Besicovich bounded sequence, T: $L_p\to L_p$ is the Dunford-Schwartz operator. We put

$$f_\infty = E(f|\mathcal{A}_\infty),$$
$$A_n(f_\infty,T) = \frac{1}{n}\sum_{k=0}^{n-1}\alpha(k)T^k(f_\infty),$$
$$f_\infty^* = \lim_{n\to\infty} A_n(f_\infty,T).$$

Lemma 3.1: Let $f_n\xrightarrow{(o)} f$, $h = \sup_n|f_n|\in L_1$, and $f^* = \lim_{n\to\infty}A_n(f,T)$. Then $A_n(f_n,T)\xrightarrow{(o)} f^*$ in L_0.

Proof: Let $h_n = \sup_{m\geq n}|f_m - f|$.

Obviously,

$$|A_n(f_n,T) - f^*| \leq |A_n(f_n,T) - A_n(f,T)| + |A_n(f,T) - f^*|$$

By theorems 1.2 and 1.4 [9], $A_n(f,T)\xrightarrow{(o)} f^*$ as $n\to\infty$ in L_0. We will prove that

$$A_n(f_n,T) - A_n(f,T)\xrightarrow{(o)} 0 \text{ as } n\to\infty \text{ in } L_0.$$

It is clear,

$$|A_n(f_n,T)\quad A_n(f,T)| = |A_n(f_n\quad f,T)| \leq \frac{1}{n}\sum_{k=0}^{n-1}|\alpha(k)||T|^k(|f_n\quad f|)\leq$$
$$b\cdot\frac{1}{n}\sum_{k=0}^{n-1}|T|^k(|f_n\quad f|)\leq b\cdot S_n(|f_n\quad f|,|T|).$$

Since $|f_n - f|\xrightarrow{(o)} 0$ and $\sup_n|f_n - f|\leq 2h\in L_1$, we obtained from Lemma 3, that $S_n(|f_n - f|,|T|)\xrightarrow{(o)} 0$ as $n\to\infty$. Therefore $A_n(f_n,T) - A_n(f,T)\xrightarrow{(o)} 0$ and $A_n(f_n,T)\xrightarrow{(o)} f^*$ in L_0.

Theorem 3.2

1) Let $\in(1,\infty)$. Then $A_n(E(f|\mathcal{A}_n),T)\xrightarrow{(o)} f_\infty^*$ in L_p as $n\to\infty$.

2) Let p = 1 and $\sup_n|E(f|\mathcal{A}_n)|\leq h\in L_1$. Then $A_n(E(f|\mathcal{A}_n),T)\xrightarrow{(o)} f_\infty^*$ in L_0.

Proof

1) According to Theorem 2 [13], $\sup_{n\geq 1}E(|f||\mathcal{A}_n)\in L_p$. Therefore by Lemma 3.1, we have $A_n(E(f|\mathcal{A}_n),T)\xrightarrow{(o)} f_\infty^*$ in L_0.

Let $\sup_{n\geq 1} E(|f||\mathcal{A}_n) = h$. Then $E(|f||\mathcal{A}_n) \leq h$ and $A_n(E(|f||\mathcal{A}_n), |T|) \leq A_n(h, |T|)$. As by the Akcoglu's theorem $\sup_{n\geq 1} A_n(h, |T|) \in L_p$, we have $\sup_{n\geq 1} A_n(E(|f||\mathcal{A}_n)|T|) \in L_p$ and $\sup_{n\geq 1} |A_n(E(f|\mathcal{A}_n), T)| \in L_p$.
Hence, $A_n(E(f|\mathcal{A}_n)T) \xrightarrow{(o)} f_\infty^*$.

2) By the Theorem 3 [13], we have

$$E(f|\mathcal{A}_n) \xrightarrow{(o)} E(f, \mathcal{A}_\infty)$$

in L_0. As $\sup_{n\geq 1} |E(f|\mathcal{A}_n| \in L_1$, by Lemma 3.1 we obtain

$$A_n(E(f|\mathcal{A}_n)T) \xrightarrow{(o)} f_\infty^*$$

in L_0.

WEIGHTED DOMINANT AND MAXIMAL INEQUALITIES

Theorem 4.1: Let p∈(1,∞) and $\mathcal{A}_n \downarrow \mathcal{A}_\infty$. Then for $f \in L_p$ the following inequality holds:

$$\| \sup_{n\geq 1} |E(A_n(f,T)|\mathcal{A}_n)| \|_{L_p} \leq bq^2 \| f \|_{L_p},$$

where $b = \sup_k|\alpha(k)|$.

Proof: Let $g = \sup_{n\geq 1} |A_n(f,T)|$. Then

$$|A_n(f,T)| \leq g, \quad \sup_{n\geq 1} |E(A_n(f,T)|\mathcal{A}_n)| \leq \sup_{n\geq 1} E(g|A_n)$$

and therefore

$$\| \sup_{n\geq 1} |E(A_n(f,T)|\mathcal{A}_n)| \|_{L_p} \leq \| \sup_{n\geq 1} E(g|\mathcal{A}_n) \|_{L_p}.$$

By the dominant inequality for submartingale [12] we have

$$\| \sup_{n\geq 1} E(g|\mathcal{A}_n) \|_{L_p} \leq q \| E(g|\mathcal{A}_1) \|_{L_p}.$$

Since the conditional expectation operator is contracting in L_p we obtain, that

$$\| E(g|\mathcal{A}_1) \|_{L_p} \leq \| g \|_{L_p}.$$

Therefore

$$\| \sup_{n\geq 1} |E(A_n(f,T)|\mathcal{A}_n)| \|_{L_p} \leq q \| \sup_{n\geq 1} |A_n(f,T)| \|_{L_p}.$$

It follows from Akcoglu's theorem that $\| \sup_{n\geq 1} |A_n(f,T)| \|_{L_p} \leq b \cdot q \| f \|_{L_p}$. Thus, for $f \in L_p$ the inequality

$$\| \sup_{n\geq 1} |E(A_n(f,T)|\mathcal{A}_n)| \|_{L_p} \leq bq^2 \| f \|_{L_p}$$

holds.

Theorem 4.2: Let $f \in L_p$, p∈(1,∞) and $\mathcal{A}_n \downarrow \mathcal{A}_\infty$. Then for any ε>0 the following inequality holds

$$\lambda\{\sup_{n\geq 1} |E(A_n(f,T)|\mathcal{A}_n|)\} \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \cdot b^p \cdot \| f \|_{L_p}^p,$$

where $b = \sup_k|\alpha(k)|$.

Proof: Let $g = \sup_{n\geq 1} |A_n(f,T)|$. Then

$$\lambda\{\sup\ |E(A_n(f,T)|\mathcal{A}_n)| \geq \varepsilon\} \leq \lambda\{\sup\ E(g|\mathcal{A}_n) \geq \varepsilon\}$$

By the maximal inequality for for submartingale $E(g|\mathcal{A}_n)$ [12] we have, that

$$\lambda\{E(g|A_n) > \varepsilon\} < \frac{1}{\varepsilon^p} \| E(g \| \mathcal{A}_1) \|_{L_p}^p.$$

Applying the contracting property of the conditional expectation operator in L_p, we obtain

$$\| E(g|\mathcal{A}_n) \|_{L_p}^p \leq \| g \|_{L_p}^p = \| \sup_{n\geq 1} |A_n(f,T)| \|_{L_p}^p \leq \| \sup_{n\geq 1} A(|f|,|T|) \|_{L_p}^p.$$

Now applying the inequality $\| \sup_{n\geq 1} A_n(|f|,|T|) \|_{L_p} \leq b\cdot q \| f \|_{L_p}$, we have

$$\lambda\{\sup_{n\geq 1} |E(A_n(f,T)|\mathcal{A}_n)| \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} b^p \cdot q^p \cdot \| f \|_{L_p}^p.$$

The following theorems can be proved analogously to theorems 4.1 and 4.2.

Theorem 4.3: Let $p\in(1,\infty)$. Then for $f\in L_p$ the following inequality holds

$$\| \sup_{n\geq 1} |A_n(E(f|\mathcal{A}_n)T)| \|_{L_p} \leq b\cdot q^2 \| f \|_{L_p},$$

where $b = \sup_k|\alpha(k)|$.

Theorem 4.4: Let $p\in(1,\infty)$. Then for any $\varepsilon>0$ and $f\in L_p$ the following inequality holds

$$\lambda\{\sup_{n\geq 1} |A_n(E(f|\mathcal{A}_n),T)| \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} b^p \cdot q^p \| f \|_{L_p}^p,$$

where $b = \sup_k|\alpha(k)|$..

MULTIPARAMETER WEIGHTED MARTINGALE ERGODIC THEOREMS

In this section we define the d-dimensional case of martingale ergodic averages and consider convergence theorems for such averages.
Let $\{a(k): k \in Z_d^+\}$ be the class of weights and

$$k = (k_1, k_2, \ldots, k_d), \boldsymbol{N} = (N_1, N_2, \ldots, N_d), |\boldsymbol{N}| = N_1 \cdot N_2 \cdot\ldots\cdot N_d, \boldsymbol{0} = (0,0,\ldots,0)$$

For $N = (N_1, N_2,\ldots, N_d)$ $N\to\infty$ means that $N_i\to\infty$ for any $i = 1,2,\ldots,d$.

Definition 5.1: [9] The sequence $\{a(k): k \in Z_d^+\}$ is called r-Besicovich if for every $\varepsilon>0$ there is a sequence of trigonometric polynomials in d variables, $\psi_\varepsilon(k)$, such that $\lim_{|N|\to\infty} \sup \frac{1}{|N|} \sum_{k=0}^{N-1} |a(k) - \psi_\varepsilon(k)| < \varepsilon$.

This class is denoted by B(r). The sequence $\alpha(k)$ is called r-bounded Besicovitch if $\{a(k)\} \in B(r) \cap l^\infty$. If $\{a(k)\} \in B(1) \cap l^\infty$ then $\{\alpha(k)\}$ is called a bounded Besicovich sequence.
Let $T_1, T_2,\ldots, T_d$ denote a family of d linear operators in L_p. We consider averages

$$A_N(\boldsymbol{T})f = \frac{1}{|\boldsymbol{N}|}\sum_{k=0}^{N} a(k)\boldsymbol{T}^k f.$$

where $f \in L_p, \boldsymbol{T}^k = T_1^{k_1} \cdot T_2^{k_2} \cdot\ldots\cdot T_d^{k_d}$
In [9] it is proved the following

Theorem 5.2: Let T = (T_1, T_2,…, T_d) denoted d Dunford-Schwartz operators. Then $A_N(T)f$ converges a.e. for every $f\in L_p$, $1<p\leq\infty$ and all bounded Besicovich sequences $\alpha(k)$.

Let, as in the Section 2 $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a monotone sequence of σ-subalgebras of Σ and $\mathcal{A}_n \downarrow \mathcal{A}_\infty$. We put $f^* = \lim_{N\to\infty} A_N(\boldsymbol{T})f, f_\infty^* = E(f^*|\mathcal{A}_\infty)$.

Theorem 5.3: Let $f\in L_p$, $1<p\leq\infty$, $\alpha(k)$ be a bounded Besicovich sequence, $\boldsymbol{N}_n = (N_1^n, N_2^n, \ldots, N_d^n) \to \infty$ as $n\to\infty$. Then

$$E(A_{N_n}(\boldsymbol{T})f|\mathcal{A}_n) \xrightarrow{(p)} f_\infty^*$$

in L_p.

The proof of Theorem 5.3 is analogous to the proof of Theorem 2.1 1).

Theorem 5.4: Let $\mathcal{A}_l \downarrow \mathcal{A}_\infty$, $f\in L_p$, $1<p\leq\infty$, $\alpha(k)$ bounded Besicovich sequence, $b = \sup|\alpha(k)|$, $\boldsymbol{N}_l = (N_1^l, N_2^l, \ldots, N_d^l) \to \infty$ as $l\to\infty$. Then

1) $\| \sup_l |E(A_{N_l}(\boldsymbol{T})f|\mathcal{A}_l)| \|_p \leq b\cdot q^{d+1} \| f \|_p$;

2) $\lambda\{\sup_l |E(A_{N_l}(\boldsymbol{T})f|\mathcal{A}_l)| \geq \varepsilon\} \leq b\cdot q^{pd}\frac{\|f\|_p^p}{\varepsilon^p}$.

Proof

1) Let $g_l = \sup_{m\geq l}|A_{N_m}(\boldsymbol{T})f|$. Then

$$|E(A_{N_l}(\boldsymbol{T})f|\mathcal{A}_l)| \leq E(\sup_{m\geq l}|A_{N_m}(\boldsymbol{T})f|\mathcal{A}_l) = E(g_l|\mathcal{A}_l)$$

and

$$\sup_l |E(A_{N_l}(\boldsymbol{T})f|\mathcal{A}_l)| \leq \sup_l E(g_l|\mathcal{A}_l). \qquad (1)$$

Therefore

$$\| \sup_l |E(A_{N_l}(\boldsymbol{T})f|\mathcal{A}_l)| \|_p \leq \| \sup_l E(g_l|\mathcal{A}_l) \|_p.$$

By the dominant inequality,

$$\| \sup_l E(g_l|\mathcal{A}_l) \|_p \leq q \, \| E(g_1|\mathcal{A}_1) \|_p.$$

Since the conditional expatiation operators is contracting in L_p, we have

$$\| E(g_1|\mathcal{A}_1) \|_p \leq \| g_1 \|_p. \qquad (2)$$

Thus

$$\| \sup_l |E(A_{N_l}(\boldsymbol{T})f|\mathcal{A}_l)| \|_p \leq q \, \| g_1 \|_p =$$
$$= q \, \| \sup_m A_{N_m}(\boldsymbol{T})|f| \, \|_p \leq q\cdot q^d \cdot b \, \| f \|_p =$$
$$= b\cdot q^{d+1} \| f \|_p.$$

2) By the inequality (1) we have

$$\lambda\{\sup_l |E(A_{N_l}(\boldsymbol{T})f|\mathcal{A}_l)| \geq \varepsilon\} \leq \lambda\{\sup_l E(g_l|\mathcal{A}_l) \geq \varepsilon\}$$

According to the maximal inequality

$$\lambda\{\sup_l E(g_l|\mathcal{A}_l) \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \| E(g_l|\mathcal{A}_l) \|_p^p.$$

Applying inequality (2), we obtain

$$\frac{1}{\varepsilon^p} \| E(g_l|\mathcal{A}_l) \|_p^p \leq \frac{1}{\varepsilon^p} \| g_1 \|_p^p \leq \frac{1}{\varepsilon^p} \| \sup_m A_{N_m}(\boldsymbol{T})|f| \, \|_p^p.$$

Since $\| \sup_m A_{N_m}(\boldsymbol{T})|f| \|_p^p \leq b^p q^{dp} \| f \|_p$, we obtain

$$\lambda\{\sup_i |E(A_{N_i}(\boldsymbol{T})f|\mathcal{A}_i)| \geq \varepsilon\} \leq b^p \cdot q^{pd} \frac{\| f \|_p^p}{\varepsilon^p}$$

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