Middle-East Journal of Scientific Research 13 (Mathematical Applications in Engineering): 64-69, 2013
ISSN 1990-9233
© IDOSI Publications, 2013
DOI: 10.5829/idosi.mejsr.2013.13.mae.99922

On L_1-Weak Ergodicity of Markov Processes

[1]*Farrukh Mukhamedov*

[1]Department of Computational and Theoretical Sciences, Faculty of Science,
International Islamic University Malaysia, 25200, Kuantan, Pahang, Malaysia

Abstract: In the present paper we investigate the L_1-weak ergodicity of nonhomogeneous discrete Markov processes with general state spaces. Note that the L_1-weak ergodicity is weaker than well-known weak ergodicity. It was known a necessary and sufficient condition for such processes to satisfy the weak ergodicity. In this paper we provide a necessary and sufficient condition for nonhomogeneous discrete Markov processes to satisfy the L_1-weak ergodicity. Moreover, as an application of the main result, we provide more concrete examples of Markov processes which satisfy the L_1-weak ergodicity

Key words: Weak ergodicity . nonhomogeneous Markov process . the Doeblin's condition

INTRODUCTION

Markov processes with general state space have become a subject of interest due to their applications in many branches of mathematics and natural sciences. One of the important notions in these studies is ergodicity of Markov processes, i.e. the tendency for a chain to 'forget' the distant past. In many cases, a huge number of investigations were devoted to such processes with countable state space [1-7]. For nonhomogeneous Markov processes with countable state space, investigation of the general conditions of weak ergodicity leads to the definition of a special subclass of regular matrices. In many papers [7-9, 11] the weak ergodicity of nonhomogeneous Markov process are given in terms of Dobrushin's ergodicity coefficient [1]. In general case, one may consider several kinds of convergence [16].

In the present paper we are going to investigate the L_1-weak ergodicity of nonhomogeneous discrete Markov processes, in general state spaces. In our study we are not going to use Dobrushin's ergodicity coefficient. We have to stress that the L_1-weak ergodicity is weaker than usual weak ergodicity (see next section). It is known [2] the a necessary and sufficient condition for the homogeneous Markov process to satisfy the weak ergodicity. In this paper, we shall provide necessary and sufficient conditions for such processes to satisfy the L_1-weak ergodicity. As application of the main result, certain concrete examples are provided. It is worth to mention that in [10] a necessary and sufficient condition was found for homogeneous Markov processes to satisfy L_1-ergodicity. Our condition recovers the mentioned condition when the processes become homogeneous.

L_1-WEAK ERGODICITY

Let (X, F, μ) be a probability space. In what follows, we consider the standard $L^1(X, F, \mu)$ and $L^\infty(X, F, \mu)$ spaces. Note that $L^1(X, F, \mu)$ can be identified with the space of finite signed measures on X absolutely continuous with respect to μ. By M we denote the set of all probability measures on X which are absolutely continuous w.r.t. μ. Recall that transition probabilities $P^{[k,m]}(x,A)$, $x\in X$, $A\in F$ $(k, n\in Z_+)$ form a non-homogeneous discrete Markov process (NHDMP) iff the following conditions are satisfied:

(i) For each k,n the function $P^{[k,n]}(\cdot,A)$ is measurable for any $A\in F$; for every $x\in X$ one has $P^{[k,n]}(x,\cdot)\in M$ and it is μ-measurable, i.e. $\mu(A) = 0$ implies $P^{[k,n]}(x,A)=0$ a.e. on X;
(ii) One has Kolmogorov-Chapman equation: for every $k\le m\le n$

Corresponding Author: Farrukh Mukhamedov, Department of Computational and Theoretical Sciences, Faculty of Science, International Islamic University Malaysia, 25200, Kuantan, Pahang, Malaysia

$$P^{[k,n]}(x,A) = \int P^{[k,m]}(x,dy)P^{[m,n]}(y,A) \quad (1)$$

In the sequel, we will deal with μ-measurable NHDMP. In this case, for each k,n such one can define a positive linear contraction operator on L^1 (resp. L^∞) denoted by $P_*^{[k,n]}$ (resp. $P^{[k,n]}$). Namely,

$$(P_*^{[k,n]}\nu)(A) = \int P^{[k,n]}(x,A)\,d\nu(x), \nu \in L^1 \quad (2)$$

$$(P^{[k,n]}f)(x) = \int P^{[k,n]}(x,dy)f(y), f \in L^\infty \quad (3)$$

From (2) it follows that (1) can be rewritten as follows $P_*^{[k,n]} = P_*^{[m,n]}P_*^{[k,m]}$, where $k \le m \le n$. Recall that if for a NHDMP $P^{[k,n]}(x,A)$ one has $P_*^{[k,n]} = (P_*^{[0,1]})^{n-k}$, then such a process becomes homogeneous and therefore, it is denoted by $P^n(x,A)$.

Definition: A NHDMP $P^{[k,n]}(x,A)$ is said to satisfy

1. The weak ergodicity if for any $k \in Z_+$ one has

$$\lim_{n\to\infty}\sup_{x,y\in X} \| P^{[k,n]}(x,\cdot) - P^{[k,n]}(y,\cdot) \|_1 = 0$$

2. The L_1-weak ergodicity if for any probability measures $\lambda, \nu \in M$ and $k \in Z_+$ one has

$$\lim_{n\to\infty} \| P_*^{[k,n]}\lambda - P_*^{[k,n]}\nu \|_1 = 0$$

3. The strong ergodicity if there exists aprobability measure μ_1 such that for every $k \in Z_+$ one has

$$\lim_{n\to\infty}\sup_{x\in X} \| P^{[k,n]}(x,\cdot) - \mu_1 \|_1 \to 0$$

4. The L_1-strong ergodicity if there exists a probability measure μ_1 such that for every $k \in Z_+$ and $\lambda \in M$ one has

$$\lim_{n\to\infty} \| P_*^{[k,n]}\lambda - \mu_1 \|_1 \to 0$$

One can see that the weak (resp. strong) ergodicity implies the L_1-weak (resp. L_1-strong) ergodicity. Therefore, it is natural to find certain necessary and sufficient conditions for the satisfaction L_1-weak ergodicity of NHDMP. So, in the paper we will deal with L_1-weak ergodicity. Note that historically, one of the most significant conditions for the weak ergodicity is the Doeblin's Condition (for homogeneous Markov processes), which is formulated as follows: there exist a probability measure ν, an integer $n_0 \in N$ and constants $0<\varepsilon<1$, $\delta>0$ such that for every $A \in F$ if $\nu(A)>\varepsilon$ then $\inf_{x\in X} P^{n_0}(x,A) \ge \delta$. This condition does not imply either the aperiodicity or the ergodicity of the process. In [8] the aperiodicity is studied by minorization type conditions, i.e. there exist a non-trivial positive measure λ and $n_0 \in N$ such that $P^{n_0}(x,A) \ge \lambda(A), \forall x \in X, \forall A \in F$. But the last condition is not sufficient for the strong ergodicity. In [10] it was introduced a variation of the above condition, i.e. Condition (C_0): there exists a non-trivial positive measure $\mu_0 \in L^1$, $\|\mu_0\|_1 \ne 0$ and for every $\lambda \in M$ one can find a sequence $\{X_n\} \subset F$ with $\mu(X \setminus X_n) \to 0$, as $n\to\infty$ and $n_0 \in N$ such that for all $n \ge n_0$ one has

$$P_*^n\lambda \ge \mu_0 1_{X_n} \quad (4)$$

where 1_Y stands for the indicator function of a set Y. Here and in what follows, for a given $B \in F$ the measure $\mu 1_B$ is defined by $\mu 1_B(Y) = \mu(Y \cap B)$ for any $Y \in F$. It has been proved that such a condition is necessary and sufficient for the L_1-strong ergodicity of homogeneous processes.

In the present paper we shall introduce a simple variation of the above condition (A_0) for NHDMP and prove that the introduced condition is a necessary and sufficient for the L_1-weak ergodicity. Note that an other direction of variation of the Doeblin's Condition has been studied in [2]. There, by means of the Dobrushin's ergodicity coefficient it has been proved a necessary and sufficient condition for the weak ergodicity.

MAIN RESULTS

In this section we are going to introduce a simple variation of the Condition (C_0).

Definition: We say that a NHDMP $P^{[k,n]}(x,A)$ given on (X, F, μ) satisfies the condition (C_1) if for each $k \in Z_+$ there exist a positive measure $\mu_k \in L^1$, $\|\mu_k\|_1 \neq 0$ and for every $\delta > 0$ and $\lambda, \upsilon \in M$ one can find a set $X_k \in F$ with $\mu(X \setminus X_k) < \delta$ and an integer $n_k \in N$ such that

$$P_*^{[k,k+n_k]}\lambda \geq \mu_k 1_{X_k}, \; P_*^{[k,k+n_k]}\upsilon \geq \mu_k 1_{X_k} \tag{5}$$

here as before 1_Y stands for the indicator function of a set Y.

Remark: In (5) without loss of generality one may assume that $\|\mu_k\|_1 < 1/2$, otherwise we replace μ_k with $\mu_k = \mu_k/2$.

Proposition 3.1: Let a NHDMP $P^{[k,n]}(x,A)$ given on (X, F, μ). Then for the following assertions

(i) $P^{[k,n]}(x,A)$ satisfies the condition (C_1);

(ii) For any $\lambda, \nu \in M$ and $k \in Z_+$ there is a sequence $\{n_i\}$ such that for all $n \geq K_\ell := \sum_{i=1}^{\ell} n_i$ ($K_0 = k$) one has

$$||P_*^{[k,n]}\lambda - P_*^{[k,n]}\nu\|_1 = \left(\prod_{i=1}^{\ell}\gamma_i\right) \times ||P_*^{[K_\ell,n]}\lambda_\ell - P_*^{[K_\ell,n]}\nu_\ell||_1 \tag{6}$$

where $\lambda_\ell, \nu_\ell \in M$ and

$$\frac{1}{2} \leq \gamma_i \leq 1 - \frac{\|\mu_{K_{i-1}}\|_1}{2}, i = 1, \ldots, \ell$$

the implication hold true: (i)⇒(ii).

Next theorem shows that the condition (C_1) is equivalent to the satisfaction of the L_1-weak ergodicity of NHDMP.

Theorem 3.2: Let a NHDMP $P^{[k,n]}(x,A)$ be given on (X, F, μ). Then the following assertions are equivalent $P^{[k,n]}(x,A)$ satisfies the condition (C_1) and one has

$$\sum_{n=1}^{\infty}\left(1 - \frac{||\mu_{k_n}\|_1}{2}\right) = \infty \tag{7}$$

for any increasing subsequence $\{k_n\}$ of N. $P^{[k,n]}(x,A)$ satisfies the L_1-weak ergodicity.

Proof: (i)⇒(ii). Then due to Proposition 3.1 there is a subsequence $\{K_\ell\}$ such that

$$||P_*^{[k,n]}\lambda - P_*^{[k,n]}\nu\|_1 = \left(\prod_{i=1}^{\ell}\gamma_i\right)||P_*^{[K_\ell,n]}\lambda_\ell - P_*^{[K_\ell,n]}\nu_\ell\|_1$$

where $\lambda_\ell, \nu_\ell \in M$. Now from (7) one gets

$$||P_*^{[k,n]}\lambda - P_*^{[k,n]}\nu\|_1 \le 2\prod_{i=1}^{\ell}(1-\frac{\|\mu_{K_{i-1}}\|_1}{2})$$

According to (7) we get the desired assertion.

Now consider the implication (ii)⇒(i). Fix 1>ε>0. Then given $k\in N$ and $\lambda,\mu_0 \in M$, (here μ_0 is a fixed measure) one has

$$||P_*^{[k,n]}\lambda - P_*^{[k,n]}\mu_0\|_1 \to 0 \ \text{ as } \ n\to\infty$$

Then there is a sequence $\{Y_n\}\subset F$ such that $\mu(X\setminus Y_n)\to 0$, as $n\to\infty$ and

$$||(P_*^{[k,n]}\lambda - P_*^{[k,n]}\mu_0)1_{Y_n}\|_\infty \to 0 \ \text{ as } \ n\to\infty$$

Therefore, there exists an $n_k\in N$ such that $\mu(X\setminus Y_{n_k})<\varepsilon$ and

$$||(P_*^{[k,k+n_k]}\lambda - P_*^{[k,k+n_k]}\mu_0)1_{Y_{n_k}}\|_\infty < \frac{\varepsilon}{2} \tag{8}$$

Now denote $\nu_k = P_*^{[k,k+n_k]}\mu_0$. Hence, from (8) we get

$$P_*^{[k,k+n_k]}\lambda \ge P_*^{[k,k+n_k]}\lambda 1_{Y_{n_k}} \ge \nu_k 1_{Y_{n_k}} - \frac{\varepsilon}{2}1_{Y_{n_k}} \ge \mu_k 1_{Y_{n_k}}$$

where

$$\mu_k = \frac{1}{2}\nu_k 1_{A_k},\ A_k = \{x\in X : \nu_k(x)\ge \frac{\varepsilon}{2}\}$$

Since v_k is a probability measure, therefore, we have $0<\|\mu_k\|_1\le 1/2$, so $1-\frac{\|\mu_k\|_1}{2}\ge\frac{3}{4}$.

Hence, this completes the proof.

Now let us consider a nonhomogeneous version of the condition (C_0). Namely, we say that a NHDMP $P^{[k,n]}(x,A)$ given on (X, F, μ) satisfies the condition (C_2) if for each $k\in Z_+$ there exists a positive measure $\mu_k\in L^1$, $\|\mu_k\|_1\neq 0$ and for every $\lambda,\upsilon\in M$ one can find a sequence $\{X_n^{(k)}\}\subset F$ with $\mu(X\setminus X_n^{(k)})\to 0$, as $n\to\infty$ and $n_0(\lambda,k)\in N$ such that for all $n\ge n_0(\lambda,k)$ one has $P_*^{[k,n]}\lambda\ge\mu_k 1_{X_n^{(k)}}$, $P_*^{[k,n]}\upsilon\ge\mu_k 1_{X_n^{(k)}}$.

From Proposition 3.1 and Theorem 3.2 we immediately see that the condition (C_2) with (7) is sufficient for the L_1-weak ergodicity. One the other hand, if NHDMP becomes homogeneous then the condition (C_2) reduces to (4), but in [10] it has been proved that the last condition (i.e. (4)) is equivalent to the L_1-strong ergodicity of the homogeneous process. Therefore, one can formulate the following:

Problem: Is the Condition (C_2) with (7) necessary for the L_1-weak ergodicity?

APPLICATIONS

In this section we provide some application of the main result for concrete cases. let us consider a countable state space NHDMP. Namely, let X = N and μ be the Poisson measure. Then NHDMP can be given in a form of stochastic matrices $\{p_{i,j}^{[k,n]}\}_{i,j\in N}$.

Theorem 4.1: Let $\{p_{i,j}^{[k,n]}\}_{i,j\in N}$ be a NHDMP. If there exists a sequence $\{\lambda_n\}_{\in N}$, $0\le\lambda_n\le 1$ satisfying

$$\sum_{n=1}^{\infty}(1-\lambda_n)=\infty \tag{9}$$

and such that for some sequence of states $\{n_k\}$

$$p_{i,n_k}^{[k-1,k]} \geq \lambda_k \quad \text{for all } i,k \in N \tag{10}$$

then the NHDMP satisfies the L_1-weak ergodicity.

Proof: Now we show that the process satisfies the condition (C_1). Indeed, for each $k \in Z_+$ we first define a measure $\mu^{(k)}$ on X as follows:

$$\mu_i^{(k)} = \begin{cases} \lambda_k, & i = n_k \\ 0, & i \neq n_k \end{cases}$$

It is clear that $\| \mu^{(k)} \|_1 \neq 0$. From (10) it follows that

$$p_{i,j}^{[k-1,k]} \geq \mu_j^{(k)}, \text{ for all } i,j \in N \tag{11}$$

Now take any $v \in M$ and each $k \in Z_+$ we put $X_k = X$, then from (11) one finds

$$P_*^{[k-1,k]} \nu \geq \mu^{(k)} \text{ for all } k \in N$$

Hence, the condition (C_1) is satisfied. So, taking into account (10), from Theorem 3.2 we get the desired assertion.

Example 4.1: Assume that the transition probability $p_{ij}^{[k,k+1]}$ is defined by

$$p_{ij}^{[k,k+1]} = q_{ij}^{(k)} \lambda_{k,j} + r_{k,i} \delta_{ij}, \quad i,j \in N, k \in N \tag{12}$$

here $\lambda_{k,j}, q_{ij}^{(k)}, r_{k,i}$ are positive numbers with the following constrains

$$\sum_{j=1}^{\infty} (q_{ij}^{(k)} \lambda_{k,j} + r_{k,i} \delta_{ij}) = 1, \text{ for all } i \in N \tag{13}$$

It is clear that $p_{ik}^{[k,k+1]} \geq \lambda_{k,k} q_{i,k}^{(k)}$. Now assume that $\inf\{q_{ik}^{(k)} : i \in N\} := \gamma_k > 0$ and

$$\sum_{k=1}^{\infty} (1 - \lambda_{k,k} \gamma_k) = \infty$$

Then one can see that $p_{ik}^{[k,k+1]} \geq \lambda_{k,k} \gamma_k$, this means that conditions of Theorem 3.2 are satisfied with $n_k = k$, $\lambda_k = \lambda_{k,k} \gamma_k$. Hence, the defined NHDMP is L_1-weak ergodic.

Example 4.2: Consider the previous example, but now we will provide more exact values of $\lambda_{k,j}, q_{ij}^{(k)}, r_{k,i}$. Define

$$r_{k,i} = \frac{1}{k+i}, \quad \lambda_{k,j} = \begin{cases} 0, & 1 \leq j \leq k-2 \text{ or } j \geq k+1 \\ \alpha_k, & j = k-1 \\ \beta_k, & j = k \end{cases} \tag{14}$$

Note that α_k, β_k will be chosen later on. Let $q_{ik}^{(k)} = \beta_k$ for all $i \in N$ and $q_{ij}^{(k)} = 0$ for every $1 \leq j \leq k-2$ and $j \geq k+1$. Now define $q_{i,k-1}^{(k)}$ from the equality (13) as follows

$$\alpha_k q_{i,k-1}^{(k)} + \beta_k^2 + r_{k,i} = 1$$

which implies that

$$q_{i,k-1}^{(k)} = \frac{1}{\alpha_k}(1 - r_{k,i} - \beta_k^2) \tag{15}$$

Now choose α_k and β_k as follows

$$\alpha_k = \frac{1}{k}, \beta_k = \sqrt{\frac{k-1}{k}}, k \in N \tag{16}$$

Then from (14)-(16) one finds $q_{i,k-1}^{(k)} = \frac{i}{k+i}$. It is clear that $\gamma_k = \beta_k$, therefore, from (14),(16) one gets

$$\sum_{k=1}^{\infty}(1 - \lambda_{k,k}\gamma_k) = \infty$$

Hence, due to Theorem 3.2 the following NHDMP defined by

$$p_{ij}^{[k,k+1]} = \begin{cases} \dfrac{\delta_{ij}}{k+i}, 1 \le j \le k-2 \text{ or } j \ge k+1 \\ \dfrac{1}{k+i}(\dfrac{i}{k} + \delta_{i,k-1}), \; j = k-1 \\ \dfrac{k-1}{k} + \dfrac{1}{k+i}\delta_{i,k}, \; j = k \end{cases}$$

satisfies the L_1-weak ergodicity.

ACKNOWLEDGMENTS

The author acknowledges the MOHE grant FRGS11-022-0170 and the Junior Associate scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

REFERENCES

1. Dobrushin, R.L., 1956. Central limit theorem for nonstationary Markov chains. I,II. Theor. Probab. Appl., 1: 65-80, 329-383.
2. Dorea, C.C.Y. and A.G.C. Pereira, 2006. A note on a variation of Doeblin's condition for uniform ergodicity of Markov chains. Acta Math. Hungar, 110: 287-292.
3. Iosifescu, M., 1972. On Two Recent Papers on Ergodicity in Nonhomogeneous Markov Chains. Ann. Math. Statist., 43: 1732-1736.
4. Johnson, J. and D. Isaacson, 1988. Conditions for strong ergodicity using intensity matrices. J. Appl. Probab., 25: 34-42.
5. Hajnal, J., 1958. Weak ergodicity in non-homogeneous Markov chains. Proc. Cambridge Phil. Soc., 58: 233-246.
6. Madsen, R.W. and P.S. Conn, 1973. Ergodic behavior for nonnegative kernels. Ann. Prob., 1: 995-1013.
7. Madsen, R.W. and D.L. Isaacson, 1973. Strongly ergodic behavior for non-stationary Markov processes. Ann. Probab., 1: 329-335.
8. Nummelin, E., 1984. General irreducible Markov chians and non-negative operators. Cambridge Univ. Press, Cambridge.
9. Paz, A., 1970. Ergodic theorems for infinite probabilistic tables. Ann. Math. Statist., 41: 539-550.
10. Sarymsakov, T.A. and G.Ya. Grabarnik, 1987. Regularity of monotonically continuous contractions acting on the von Neumann algebra. Dokl. Akad. Nauk UzSSR, 5: 9-11.
11. Tan, Ch.P., 1996. On the weak ergodicity of nonhomogeneous Markov chains. Statis. Probab. Lett., 26: 293-295.