

Single Polygon Counting on Cayley Tree of Order 4: Generalized Catalan Numbers

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Abstract: We showed that one form of generalized Catalan numbers is the solution to the problem of finding different single component containing single fixed root for Cayley tree of order 4. The upper and lower bounds are given for semi-infinite Cayley tree of order 4.

Key words: Cayley tree . contour method . catalan numbers . ratio of gamma function . asymptotic estimate

INTRODUCTION

In network theory, there are a lot of interesting problem of counting problems. In [1] it is showed that the matrix chain multiplication problem can be transformed (or reduced) into the problem of partitioning a convex polygon into non-intersecting triangles where the solution to this problem is exactly Catalan numbers [2]. The number of binary trees with n nodes is also Catalan numbers. Beside these application problems, there are many more counting problem related to Catalan numbers [2].

In our previous paper [3, 4], we have shown that the problem of finding the number of m vertices single connected component in semi-infinite Cayley tree of order 2 (for details of Cayley tree, one can refer [5]) and order 3. The solution is also ordinary Catalan numbers [11] and the generalized one. The motivation of finding such an estimate of these numbers were given in the same paper [7], that is to solve a combinatoric problem in contour method [6-8] for lattice models. Despite the fact that the problem on Catalan numbers itself regarding the identity and properties were extensively being studied, we restrict ourself only to the problem of finding a suitable estimate such that, the Catalan numbers always

$$c_n < \frac{a^n}{n^{3/2}}$$

where $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$ is to be determined as the center of our problem. In general, Catalan numbers only correspond to the semi infinite Cayley tree, we are also interested in another sequence which is associated to the full graph. In this paper, we would like to extend our study from Cayley tree of order 2 and 3 to order 4. We also employ Gamma functions to express Catalan numbers. We first find an expression for the similar sequence for semi-infinite Cayley tree of order 4, i.e. exactly a generalized Catalan numbers.

Definition 1: A semi infinite Cayley tree of order 4, denoted as Γ_{semi}^4 , is a graph with no cycles, each vertex emanates 5 edges except the root denoted as x^0 which emanates only 4 edges (Fig. 1). We denote the set of all vertices as V and the set of all edges as L , i.e., $\Gamma_{semi}^4 = (V, L)$.

METHODOLOGY

In this section, we are going to defined our problem on the semi-infinite Cayley tree of order 4 is one form of the well known generalized Catalan numbers.

Definition 2: Let C_n be the number of vertices connected component containing a fixed root x^0 , subset to V , vertices of $\Gamma_{semi}^4 = (V, L)$.

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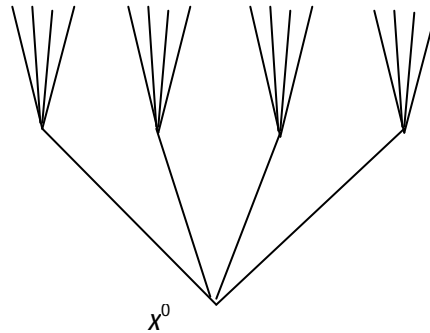


Fig. 1: Semi-infinite cayley tree of order 4

Similar as in [3], we adopted the phrase “single polygon” from the same problem but defined on integer lattice [9]. We will use similar argument as in [7] to prove following results.

RESULTS AND DISCUSSION

Theorem 1: C_n can be written in nonlinear recursion as:

$$C_n = \sum_{i+j+k+l=n-1} C_i C_j C_k C_l, \quad C_0 = 1 \quad (1)$$

where, $i, j, k, l \in \mathbb{N} \cup \{0\}$.

Proof: We divide the problem of finding number of vertices which containing a root into 4 parts consist of i, j, k and l number of vertices connect to the root, i.e. the successor of the root. The total combination is the product of the number of all successors $C_i C_j C_k C_l$. Then is sum over $C_i C_j C_k C_l$ for all $i+j+k+l = n-1$. We define $C_0 = 1$ is simply a result from observation.

The equation (1) can also be written as

$$C_n = \sum_{i=0}^n \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} C_i C_j C_k C_{n-i-j-k-1},$$

by replacing by and single sum by tree sums. The estimate for the problem defined on semi infinite Cayley tree of order 4 is straightforward after we obtain explicit form of as follows :

Theorem 2: Let C_n is the Catalan numbers,

$$C_n = \frac{1}{3n+1} \binom{4n}{n}, \quad n=0,1,2,3,\dots \quad (2)$$

Proof: Using a generating function

$$u = \sum_{i=0}^{\infty} C_i x^i = 1 + x + 4x^2 + \dots \quad (3)$$

we can obtain following relationship using (1)

$$u^4 = \sum_{i=0}^{\infty} C_i x^i \sum_{j=0}^{\infty} C_j x^j \sum_{k=0}^{\infty} C_k x^k \sum_{l=0}^{\infty} C_l x^l = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} C_i C_j C_k C_l x^{i+j+k+l} = \sum_{i=0}^{\infty} C_{i+1} x^{i+1}$$

Multiplying both sides by x ,

$$xu^4 = u - 1$$

The unique solution to this equation, by Lagrange inversion formula is

$$C_n = \frac{1}{3n+1} \binom{4n}{n}.$$

Theorem proved.

The first few terms of the sequence are given as (A002293, Integer Sequence Database)

$$1, 1, 4, 22, 140, 969, 7084, 53820, \dots$$

Using (1) and (2), one could establish following identities:

$$\begin{aligned} \frac{1}{3n+1} \binom{4n}{n} &= \sum_{i+j+k+l=n-1} \frac{1}{3i+1} \binom{4i}{i} \frac{1}{3j+1} \binom{4j}{j} \frac{1}{3k+1} \binom{4k}{k} \frac{1}{3l+1} \binom{4l}{l} \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} \sum_{k=0}^{n-1-i-j} \frac{1}{3i+1} \binom{4i}{i} \frac{1}{3j+1} \binom{4j}{j} \frac{1}{3k+1} \binom{4k}{k} \\ &= \frac{1}{3(n-i-j-k)+1} \binom{4(n-i-j-k)}{n-i-j-k} \end{aligned}$$

From result above, we can express C_n in terms of Gamma Functions:

Corollary 1: C_n can be expressed as:

$$C_n = \sqrt{\frac{2}{27\pi}} \left(\frac{256}{27}\right)^n \frac{\Gamma(n+\frac{3}{4})\Gamma(n+\frac{2}{4})\Gamma(n+\frac{1}{4})}{\Gamma(n+\frac{1}{3})\Gamma(n+1)\Gamma(n+\frac{2}{3})} \quad (4)$$

Proof: It is not difficult to deduce the recursion from theorem above:

$$C_{n+1} = \frac{4(4n+3)(4n+2)(4n+1)}{(3n+4)(3n+3)(3n+2)} C_n = \left(\frac{256}{27}\right) \frac{(n+3/4)(n+2/4)(n+1/4)}{(n+4/3)(n+1)(n+2/3)} C_n = \left(\frac{256}{27}\right)^{n+1} \frac{(n+3/4)(n+2/4)(n+1/4) \dots \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4}}{(n+4/3)(n+1)(n+2/3) \dots \frac{4}{3} \cdot 1 \cdot \frac{2}{3}} C_0$$

Therefore, one can expand C_n to C_0 and introducing $\frac{\Gamma(\frac{3}{4})\Gamma(\frac{2}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{3})\Gamma(1)\Gamma(\frac{2}{3})}$, we can obtain C_n in the form of Gamma function.

An estimate for the is obtained as follows:

Corollary 2: The inequality of is given as

$$\left(\frac{256}{27}\right)^{n-1} \frac{1}{n^{3/2}} < C_n < \sqrt{\frac{2}{27\pi}} \left(\frac{256}{27}\right)^n \frac{1}{n^{3/2}} \quad (5)$$

for $n > 0$.

Proof: From an elegant inequality proven by Wendel [10], i.e.

$$\frac{\Gamma(x+s)}{\Gamma(x)} < x^s$$

where $x > 0$ and s is a real constant such that $0 < s < 1$. Then, we can show directly that

$$\frac{\Gamma(n+\frac{3}{4})\Gamma(n+\frac{2}{4})\Gamma(n+\frac{1}{4})}{\Gamma(n+\frac{1}{3})\Gamma(n+1)\Gamma(n+\frac{2}{3})} = \frac{\Gamma(n+3/4)\Gamma(n+2/4)\Gamma(n+1/4)}{n(n+\frac{1}{4})(n+\frac{2}{3})\Gamma(n)\Gamma(n+2/3)} < \frac{(n+\frac{1}{3})^{\frac{5}{12}} n^{\frac{6}{12}} (n+\frac{2}{3})^{\frac{7}{12}}}{n(n+\frac{1}{4})(n+\frac{2}{3})} < \frac{1}{n^{3/2}}$$

From (4) and inequality above, one can show immediately the right hand side of the inequality (5)

The left hand side, we are going to prove the assetion using mathematical induction. For $n=1$, $C_0 = 1$, the equality holds. Suppose that $C_n \geq \left(\frac{256}{27}\right)^{n-1} \frac{1}{n^{3/2}}$ and multiply both side by $\frac{4(4n+3)(4n+2)(4n+1)}{(3n+4)(3n+3)(3n+2)}$,

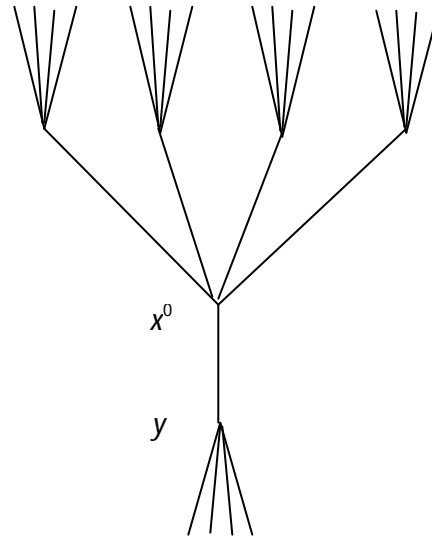


Fig. 2: A homogenous graph of Cayley tree of order

$$\frac{4(4n+3)(4n+2)(4n+1)}{(3n+4)(3n+3)(3n+2)} C_n \geq \left(\frac{256}{27}\right)^{n-1} \frac{1}{n^{3/2}} \frac{(n+3/4)(n+2/4)(n+1/4)}{(n+4/3)(n+1)(n+2/3)}$$

$$C_{n+1} > \left(\frac{256}{27}\right)^n \frac{1}{(n+1)^{3/2}}$$

Hence the result is proven.

Definition 3: Let D_n be the number of n vertices connected component containing a fixed root x^0 , subset to V , vertices of $\Gamma^4 = (V, L)$ which is Cayley tree of order 4 (Fig. 2).

Theorem 3: D_n can be written in nonlinear recursion of C_n as:

$$D_n = \sum_{r=1}^n C_r C_{n-r} \quad (6)$$

for $n > 0$.

Proof: We decompose the problem of finding the number of connected component of number of vertices containing a root x^0 into counting (i) r number of vertices which containing x^0 , i.e. C_r and (ii) $n-r$ number of vertices which containing a root y (Fig. 2) which is another successor of x^0 , i.e. C_{n-r} . Since the former C_r must always count x^0 , it should range from 1 to n . The total D_n is then the sum of all $C_r C_{n-r}$ where r range from 1 to n .

This formula lead to the connection between the sequence C_n ' and D_n 's'. The first few terms of D_n , which is found in OEIS A196678 [10], is listed as follows:

$$1, 5, 30, 200, 1425, 11110, \dots$$

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