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Total Resolvability in Graphs

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Abstract: A set of vertices W is called a weak total resolving, simply written as WTR-set, if for every pair of distinct vertices x,y in G with $x \in V(G)\backslash W$ and $y \in W$, there is a vertex $w \in W\backslash\{y\}$ such that $d(x,w) \neq d(y,w)$. A set of vertices W is called a strong total resolving, written as STR-set, if for every pair of distinct vertices x,y in G, there is a vertex w in W such that $d(x,w) \neq d(y,w)$ for $x,y \neq w$. The cardinality of a minimum WTR-set and a minimum STR-set is called the weak total metric dimension and strong total metric dimension of G, denoted by $\beta_{wt}(G)$ and $\beta_{st}(G)$, respectively. In this paper, we introduce total metric dimension of graphs and study its relationship with metric dimension and related parameters. We give some realizable results and the maximum order of a connected graph G in terms of diameter and weak total metric dimension of G has also been investigated.

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Key words: Metric dimension . weak total metric dimension . strong total metric dimension . fault-tolerant metric dimension . fault-tolerant partition . square grid

INTRODUCTION

Metric dimension is a parameter that has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry [1, 3], robot navigation [15], combinatorial optimization [19] and sonar and coast guard Loran [20], to name a few. A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. As described in [1, 3], the structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations.

We refer [5] for general graph theoretic notations and terminology not described in this paper. We consider simple connected graphs G with vertex set V(G) and edge set E(G). Motivated by the problem of uniquely determining the location of an intruder in a network, Slater [20] introduced the concept of metric dimension which was also independently studied by Harary and Melter [8]. Applications of this invariant to the navigation of robots in networks are studied by Khuller *et al*. [15], to chemistry by Chartrand *et al*. [3] and to problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures are studied by Melter and Tomescu [16]. Gary and Johnson [7] noted that to determine the metric dimension of a graph is NP-hard, however, its explicit construction was given by Khuller *et al*. [14]. For more results about the notion of metric dimension and its applications, we refer to a nice survey by Saenpholphat and Zhang [3, 6, 8-13, 17, 18].

The distance, d(u,v), between two vertices u and v in G is the minimum number of edges in a u-v path. The diameter of G, denoted by D, is the largest distance between two vertices in V(G). A vertex w resolves two vertices u and v if $d(u,w) \neq d(v,w)$. A vertex set $W \subseteq V(G)$ is said to be a resolving set for G if for every two distinct vertices u and v in V(G), there is a vertex w in W that resolves u and v. The minimum cardinality of a resolving set of G is called the metric dimension of G and is denoted by $\beta(G)$. A resolving set of order $\beta(G)$ is called a metric basis of G [3].

The code, $c_W(v)$, of a vertex $v \in V(G)$ with respect to a set $W = \{w_1,\ldots,w_k\}$ is the k-tuple $(d(v,w_1),d(v,w_2),\ldots,d(v,w_k))$. Equivalently, the set W is a resolving set if for every two vertices v and w in V(G), we

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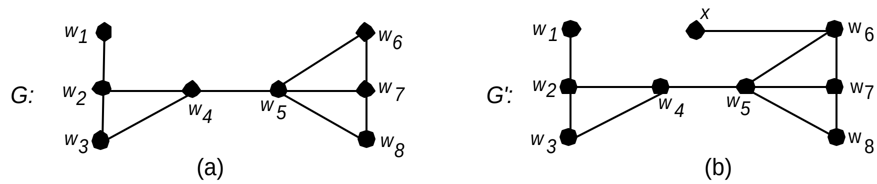

Fig. 1: (a) Graph G with resolving deficiency, (b) The resulting graph G′

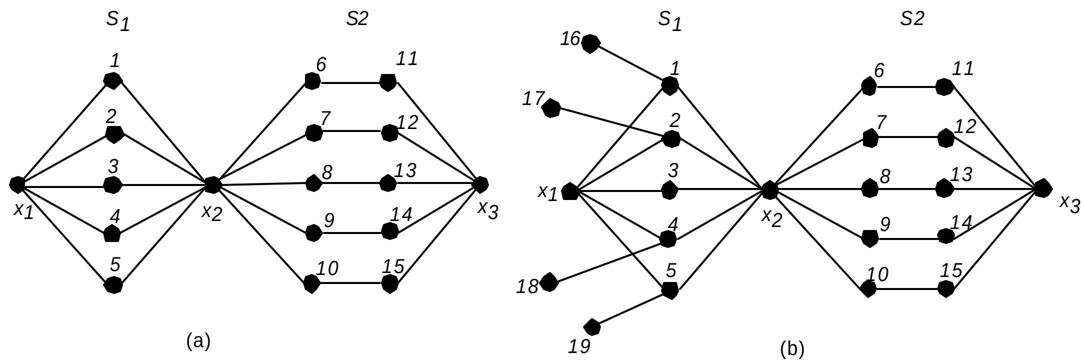

Fig. 2: (a) Graph with resolving deficiency, (b) Graph after removing the resolving deficiency

have $c_W(v) \neq c_W(w)$. From the definition of metric dimension, it can be observed that the property of a given set W of vertices of a graph G to be a resolving set of G can be verified by investigating only the vertices of V(G)\W. This is because every vertex $w \in W$ is the only vertex of G whose distance from w is 0. A resolving set W for G is called a weak total resolving, simply written as WTR-set, if for every pair of distinct vertices x,y in G with $x \in V(G)\backslash W$ and $y \in W$, there is a vertex $x \in W\backslash\{y\}$ such that $d(x,w) \neq d(y,w)$. It is clear from the definition that if V(G)\W is an empty set, then the set of vertices V(G) is a WTR-set. The cardinality of a minimum WTR-set is called the it weak total metric dimension of G, denoted by $\beta_{wt}(G)$. A WTR-set of cardinality $\beta_{wt}(G)$ is called a weak total metric basis (WTMB) of G. For a path P_n and a complete graph K_n of order n, $\beta_{wt}(P_n) = 2$ and $\beta_{wt}(K_n) = n$.

A resolving set for G is called a strong total resolving, written as STR-set, if for every pair of distinct vertices x,y in G, there is a vertex w in W such that $d(x,w) \neq d(y,w)$ for $x, y \neq w$. The minimum cardinality of an STR-set is called the strong total metric dimension of G, denoted by $\beta_{st}(G)$. An STR-set for G of cardinality $\beta_{st}(G)$ is called a strong total metric basis (STMB) of G. STR-set may or may not exist for a graph G. For example, STR-set does not exist for a complete graph on at least three vertices. We say that a graph G is totally resolved if there exists an STR-set for G.

In a graph G, if WTR-set exists but there is no subset of V(G) which admits the condition of STR-set, then we say that there is a deficiency to resolve the graph G totally and is called the resolving deficiency in G. However, this resolving deficiency can be removed by adding exactly one pendent vertex with each of those vertices of a resolving set which do not admit the condition of STR-set. The number of vertices added in G to remove the resolving deficiency is called the resolving deficiency number of G, denoted by $\eta(G)$ and the vertices used for this purpose are called the supporting vertices. After removing the resolving deficiency from G, the resulting graph will be denoted by G′ and we say that the graph G′ has strong metric dimension $\beta_{st}(G')$ with resolving deficiency number $\eta(G)$.

Example 1.1: Consider the graph G shown in Fig. 1a. One can see that for the graph G, the set $\{w_1, w_6\}$ is a minimum resolving set and the set $\{w_1, w_6, w_8\}$ is a minimum WTR-set. But, there is no subset of V(H) which admits the condition of STR-set because the vertices w_6 and w_8 have the same distances to all the vertices of G. So there is the resolving deficiency in G. However, if we use the supporting vertex x (Fig. 1b), then this deficiency has been removed from G and the set $\{x, w_1, w_6\}$ is a minimum STR-set for G′. Hence $\beta(G) = 2$, $\beta_{wt}(G) = 3$ and $\beta_{st}(G') = 3$ with $\eta(G) = 1$.

Elements of metric basis were referred to as censors in an application [2]. In an application, if one of the censors does not work properly, we will not have enough information to deal with the intruder (fire, thief, etc). Consider a graph G with two parts connected by a single vertex shown in Fig. 2a.

One can assume G as a model of a housing complex with two sectors S_1 and S_2 of parallel housing rows. Suppose that, for the sake of the security of the complex from intruder, the complex management wants to place the smallest number of censors (detecting devices for arms) in the complex in such a way that all the locations in the complex are uniquely detected by their distances to the censors. Clearly, it is shown in Fig. 2a that any four locations from exactly four parallel housing rows in each sector is a best selection of smallest number of locations to place the censors. With out loss of generality, we assume that the censors are placed at the locations 1,2,4 and 5 in the sector S_1 and at the locations 6,8,14 and 15 in sector S_2. This placement of the censors will assure the security of the whole complex only if all the censors work properly and the collection S = {1,2,4,5,6,8,14,15} is called a resolving set for G. Now the problem is that if one of the censors or any two censors stop detecting the intruder due to any impenetrable problem, then how to assure the safety of the insecured pairs of locations of the complex? For example, if the censor placed at location 2 stops working, then the location 2 and 3 can not uniquely detected by their distances to any of the remaining censors or if the censors placed at the locations 8 and 15 are collapsed, then the location 8 and 9,8 and 10,9 and 10, 13 and 14, 14 and 15, 13 and 15 all have the same distances to all the remaining censors and hence are not uniquely detected by their distances. In the context of resolvability, we say that the collection S resolves the graph G but not totally resolves G. The solution of this kind of problems is the origin of introducing the concept of total resolvability in graphs.

Note that, if we place two more censors at the locations 3 and 7, then the collection T = $S\cup\{3,7\}$ of censors assures that every pair of locations uniquely detected by their distances to some censors even any of the censors does not work and is called a WTR-set for G. But, still there is a deficiency in uniquely determining some pairs of location where the censors placed. This deficiency is that if the censors placed at locations, say 2 and 5, stopped working simultaneously, then the locations 2 and 5 are insecured and cannot be uniquely determined by their distances to any of the remaining censors. This kind of problem happen not only with the censors placed at the locations 2 and 5, but also with the censors placed at the locations {1,2,4,5} and the locations {6,8,14,15}. In fact, this is the resolving deficiency in G. However, this deficiency can be removed by attaching exactly one censor to each of the censors placed at locations {1,2,4,5,6,8,14,15}. Then the resulting collection U = $S\cup\{9,10,11,13,16,17,18,19\}$ completely assures the safety of the complex. In the context of resolvability, we call the collection U, an STR-set for G with the supporting vertices 16, 17, 18 and 19 (Fig. 2b).

MAIN RESULTS

In this section, we study the relationship of a WTR-set and an STR-set with a resolving and a fault-tolerant resolving set. Also, we give some realizable results. In the case of WTR-set, the following result holds.

Lemma 2.1: A resolving set W for a graph G is a WTR-set if and only if the code of every vertex $w\in W$ differ by at least two coordinates from the code of each vertex $v\in V(G)\backslash W$.

Proof: Suppose that W is a WTR-set for G. Contrarily suppose that the code of a vertex, say w_i, of W and a vertex $y\in V(G)\backslash W$ differ by only one coordinate. Since the code $c_W(w_i)$ of w_i has 0 at ith place, so the codes $c_W(w_i)$ and $c_W(y)$ differ by ith coordinate only, a contradiction.

Conversely, for every $x\in W$ and every $y\in V(G)\backslash W$, the codes $c_W(x)$ and $c_W(y)$ differ by at least two coordinates, which means that W a WTR-set.

Two vertices u and v in a graph G are said to be distance similar if $d(u,w)=d(v,w)$ for all $w\in V(G)\backslash\{u,v\}$. A subset U of V(G) is called a distance similar set if every two vertices in U are distance similar. By all of the above definitions, the following remark and proposition holds:

Remark 2.2:

(a) If u and v are distance similar vertices in G and W be any resolving set for G, then $u\in W$ or $v\in W$. Moreover, if $u\in W$, then $(W\backslash\{u\})\cup\{v\}$ also resolves G.

(b) If U be a distance similar set of cardinality l, then any resolving set for G contains l-1 elements of U. Any WTR-set for G contains all elements of U.

(c) A graph G has non zero resolving deficiency if and only if there exist a distance similar set in G.

(d) An STR-set for G exists if and only if no two vertices of the graph are distance similar.

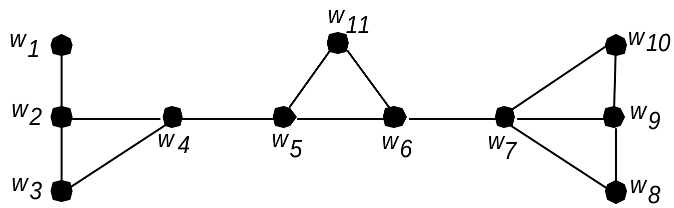

Fig. 3: The graph G

Proposition 2.3: A WTR-set W for a graph G is an STR-set if every pair x,y of vertices in W is resolved by some vertex of W other than x and y.

Now if a graph G contains distance similar vertices then by Remark 2.2(d), STR-set does not exist for G. So there is a resolving deficiency in G and in order to remove this resolving deficiency, η(G) number of supporting vertices are needed. Thus we have the following proposition:

Proposition 2.4: Let G be a graph. If there are r distance similar sets $U_1, U_2,\ldots,U_r$ in G, then $\beta_{st}(G') \geq \sum |U_i| + \beta(G) - r$, where G' is the graph obtained from G after removing the resolving deficiency. Moreover, this bound is sharp.

Proof: Since every resolving set contains $\sum |U_i| - r$ vertices from r distance similar sets $U_1, U_2,\ldots,U_r$, by Remark 2.2(b). So, at least $\sum |U_i| - r$ supporting vertices required to remove the resolving deficiency for a resolving set of cardinality β(G) to be admitted the condition of STMB of the graph G'. Further, if we let G = K_n (complete graph on n vertices), then $\beta_{st}(G') = 2n - 2$ with $\eta(G) = n - 1$.

A resolving set W of a graph G is said to be fault-tolerant (simply written as FTR-set) if W\{w}, for each $w \in W$, is also a resolving set for G. The fault-tolerant metric dimension of G is the minimum cardinality of an FTR-set, denoted by $\beta'(G)$. An FTR-set of order $\beta'(G)$ is called a fault-tolerant metric basis (FTMB) [12]. From the definitions of FTR-set and WTR-set, following lemma follows:

Lemma 2.5: A WTR-set W for a graph G is FTR if and only if every pair x,y of vertices in G such that $x,y \notin W$ is resolved by at least two vertices of W.

Proof: Suppose that W is an FTR-set for G. Assume contrarily that two vertices x and y in G are resolved by only one vertex w of W, then W\{w} is not a resolving set since x and y have same codes with respect to W\{w}, a contradiction.

Now suppose that every pair of vertices x,y in G, such that $x,y \notin W$ is resolved by at least two vertices of W. Since W is a WTR-set, W\{w}, for each w in W, is also a resolving set for G, by definition.

Example 2.6: Consider the graph G given in Fig. 3. Let $W_1 = \{w_1, w_8, w_{10}\}$. Then the codes of the vertices of G with respect to W_1 are:, which implies that W_1 is a WTR-set for G. But, W_1 is not an FTR-set because $W_1 \backslash \{w_1\}$ does not resolve G. If we add the vertex w_3 into W_1, then the set $W_2 = W_1 \cup \{w_3\}$ is a WTR-set as well as anFTR-set for G. Also, there is no 2-element subset of V(G) which weakly resolves all the vertices of G and there is no FTR-set of cardinality less than 4 for G, which implies that $\beta_{wt}(G) = 3$ and $\beta'(G) = 4$.

By Lemmas 2.1 and 2.5, we have the following result:

Lemma 2.7: A resolving set W for a graph G is an STR-set if and only if (i) W is a WTR-set for G and (ii) for every pair x,y of vertices in W, there exists a $w \notin \{x,y\}$ in W which resolves x and y.

Proof: Suppose W is an STR-set. Then, by definition, for every pair vertices in G, in particular $x \in V(G) \backslash W$ and $y \in W$, there is a vertex w in W other than y such that $d(x,w) \neq d(y,w)$, which implies that W is a WTR-set. Also, every pair of vertices in W is resolved by some other vertex of W, which implies (ii).

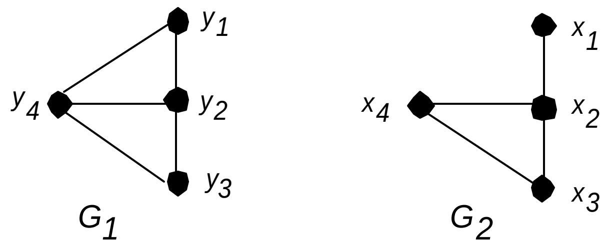

Fig. 4: Graphs to be identified

Conversely, if (i) and (ii) hold, then by Proposition 2.3, W is an STR-set.

Now we show that every pair a,b of positive integers with $a \geq k$ and $b = a+k$ $(k>0)$ is realizable as the metric dimension and weak total metric dimension some connected graphs.

Theorem 2.8: For any triplet (a,b,k) of positive integers with $a \geq k$ and $b = a+k$, there exists a graph G such that $\beta(G) = a$ and $\beta_{wt}(G) = b$. In particular, $\beta_{wt}(G) - \beta(G) = k$.

Proof: We have the following two cases:

Case 1: (When a = k) For k = 1, consider a path P_{a+b} of length a+b. Then, clearly, $\beta(G) = a$ and $\beta_{wt}(G) = b$.

For $k \geq 2$, attach the graph G_1 shown in Fig. 4 with the path P_{a+b} by identifying the vertex y_4 of G_1 and the vertices $v_1, v_2, \ldots, v_a$ of P_{a+b}. Call the resulting graph G. Note that, there are a copies of G_1 in G. Call the vertices of the ith copy of G_1, $y_1^i, y_2^i, y_3^i, y_4^i (1 \leq i \leq a)$. Then the vertex y_1^i from each copy of G_1 participates to resolve the graph G and the vertices y_1^i and y_3^i from each copy of G_1 participate to weakly resolve G, which implies that $\beta(G) = a$ and $\beta_{wt}(G) = 2a = b$.

Case 2: (When a>k) Consider the graph G_2 shown in Fig. 4. Make an identification graph $G_3 = G_3[G_2, P_{a+b}, x_4, v_1]$ by identifying $x_4 \in G_2$, $v_1 \in P_{a+b}$ and $x_4 = v_1 \in G_3$. Again identify graphs G_1 and G_3 by $G_4 = G_4[G_1, G_3, y_4, v_{a+b}]$, where $y_4 \in G_1$ and $v_{a+b} \in G_3$. Let $a - k = r \geq 1$. Now, identify $y_4 \in G_1$ with k-1 vertices of the path in graph G_4. Also, identify $x_4 \in G_2$ with r-1 vertices of the path in graph G_4. Call the resulting graph G. Note that, there are k copies of G_1 in G. Call the vertices of the ith copy of G_1, $y_1^i, y_2^i, y_3^i, y_4^i (1 \leq i \leq k)$. Also, there are r copies of G_2 in G. Call the vertices of the ith copy of G_2, $x_1^i, x_2^i, x_3^i, x_4^i (1 \leq i \leq r)$. Then the vertex y_1^i from each copy of G_1 and the vertex x_1^i from each copy of G_2 participate to resolve the graph G. The vertices y_1^i and y_3^i from each copy of G_1 and the vertices x_1^i from each copy of G_2 participate to weakly resolve the graph G, which implies that $\beta(G) = r + k = a$ and $\beta_{wt}(G) = r + 2k = b$.

The following result shows that every pair a,b of positive integers with $a \geq k + 2 (a \neq k + 3)$ and $b = a+k$ $(k \geq 0)$ is realizable as the weak total metric dimension and fault-tolerant metric dimension of some connected graphs.

Theorem 2.9: For every pair (a,b) of positive integers and for $k \geq 0$ with $a \geq k + 2 (a \neq k + 3)$ and $b = a+k$, there exists a graph G such that $\beta_{wt}(G) = a$ and $\beta'(G) = b$. In particular, $\beta'(G) - \beta_{wt}(G) = k$.

Proof: We have the following two cases:

Case 1: When k = 0, Then $a = b \geq 2$. For a = 2, consider a path P_{a+b} of length a+b, then it is straight forward to see that $\beta'(P_{a+b}) = \beta_{wt}(P_{a+b}) = a$.

Now for $a = b > 2$, attach b-1 pendent vertices to the vertex v_1 of the path P_{a+b}. Call the resulting graph G. Then the b-1 pendent vertices and $v_{a+b} \in P_{a+b}$ form a minimum WTR-set as well as a minimum FTR-set for G. Thus $\beta'(G) = \beta_{wt}(G) = a = b$.

Case 2: When k≥1. Consider two graphs G_1 and G_2 shown in Fig. 4. Make an identification graph $G_3 = G_3[G_2, P_{a+b}, x_4, v_1]$ by identifying $x_4 \in G_2$, $v_1 \in P_{a+b}$ and $x_4 = v_1 \in G_3$. Again identify graphs G_1 and G_3 by $G_4 = G_4[G_1, G_3, y_4, v_{a+b}]$, where $y_4 \in G_1$ and $v_{a+b} \in G_3$. Make an identification graph G_5 by identifying $x_4 \in G_2$ with k-1 vertices of G_4. Let $a - k = r \geq 3$. Make the graph G by attaching r-2 pendent vertices to $v_2 \in G_5$. The graph G_2 is attached with k vertices of path, so k vertices from these graphs, two from G_1, r-2 pendent vertices form a minimum WTR-set, which implies that $\beta_{wt}(G) = a$. Similarly, 2k vertices from G_2, two vertices from G_1 and r-2 pendent vertices form a minimum FTR-set for G, which implies $\beta'(G) = b$.

A graph G has fault-tolerant metric dimension equal to the order of G if G is a complete graph. So by Theorem 4, $\beta_{wt}(G) \leq n$ and minimum weak total metric dimension is 2 which exists for path but path is not the only graph whose weak total metric dimension is 2 because P_5 with a pendent vertex at v_3 also has a weak total metric dimension 2. Thus $2 \leq \beta_{wt}(G) \leq n$.

Let C_n denotes the cycle on n ≥3 vertices. Two vertices u and v of C_n are antipodal if $d(u,v) = \frac{n}{2}$. Note that no two vertices are antipodal in an odd cycle. It was shown that $\beta(C_n) = 2$. Moreover, two vertices resolve C_n if and only if they are not antipodal [14]. The following result gives the weak and strong total metric dimension of C_n.

Proposition 2.10: For all n≥3 and $n \neq 4$, $\beta_{wt}(C_n) = 3$. Moreover, the set W of three vertices forms a WTMB for C_n if and only if no two vertices of W are antipodal. Further, $\beta_{st}(C_n) = 3$ for all n≥5 and $n \neq 6$.

Proof: It is easy to see that $\beta_{wt}(C_n) \leq 3$. Since $\beta(C_n) = 2$ so by Theorem 2.8, $\beta_{wt}(C_n) \geq 3$.

Let W be a resolving set for C_n which consists of three vertices of C_n. If no two vertices in W are antipodal then by Lemma 2.1, W is a WTR-set. On the other hand, suppose that W is a WTR-set for C_n and we will discuss two cases.

Case 1: (n is odd) In this case, the result is obvious since no two vertices are antipodal in an odd cycle.

Case 2: (n is even) Suppose that W has two antipodal vertices of C_n. Then code of every x∈W and y∈V(C_n)\W differ by only one coordinate, so by Lemma 2.1, W is not a WTR-set, a contradiction.
The proof for the strong total metric dimension of C_n is similar as the proof of weak total metric dimension of C_n.

From Proposition 2.10, we notice that if we have a WTMB B_{wt} for C_n, then we get metric basis B by removing any vertex from B_{wt}. On the other hand, by adding one vertex into B, which is not antipodal with any element of B, we get B_{wt}. It means that every WTMB B_{wt} for C_n contains metric basis B as a proper subset. This suggests the following question: for each WTMB B_{wt} of a nontrivial connected graph G, does there exist a metric basis B of G such that $B \subset B_{wt}$? This question has negative answer in general. For this, we consider a rectangle in a square grid, defined below, whose metric dimension is 2 and B consists of endpoints of one side of rectangle [16]. But, its weak total metric dimension is 4 and B_{wt} consists of finitely many sets of four points not necessarily containing the metric basis B.

The graph having vertex set V = Z^2 and edge set $E = \{\{u,v\}: u - v \in \{(0,\pm 1),(\pm 1,0)\}\}$ determined by d_4 metric is called the square grid, denoted by (Z^2, ε_4) where $d_4((i,j),(i',j')) = |i - i'| + |j - j'|$ for any two vertices in Z^2. (d_4 metric is also referred to as city block distance.) The index 4 is appropriate because it represents number of points at a distance one from a given point with respect to d_4 metric. The set of vertices $(i,j) \in Z^2$ with |i|≤n and |j|≤m is called a rectangle, denoted by $R_{n,m}$. Any subset of (Z^2, ε_4) is called an image, denoted by | [16].

Lemma 2.11: In a rectangle $R_{n,m}$, the endpoints of the sides of an image | (square or rectangle) whose at least two sides are lying on the sides of $R_{n,m}$ form a WTR-set for $R_{n,m}$.

Proof: Let u and v be any two vertices of $R_{n,m}$. For fixed j;-m<j<m, we consider a line wx in $R_{n,m}$ with w = (n,j), x = (-n,j). It is clear that w and x do not form a resolving set for $R_{n,m}$ since any two vertices which are symmetric with respect to the line wx have equal city block distance to w and x. But they separates the vertices on the horizontal lines and on the lines with slopes±1. Let y = (n,k); $-m \leq k \leq m$ and $k \neq j$, be any vertex collinear to w. We note that if u and

v are symmetric with respect to the line wx then $|d(u,y)-d(v,y)|<2\tau$ where τ is distance between u and line wx. This shows that three vertices w, x and y form a resolving set for $R_{n,m}$.

Now if we add a vertex $z=(-n,k)$; $-m\le k\le m$ and $k\ne j$, into the set {w,x,y}, then every pair of vertices in $R_{n,m}$ is resolved by at least two vertices of {w,x,y,z}. So, by Lemma 2.1, the vertices w,x,y and z form a WTR-set for $R_{n,m}$ and they are the endpoints of the sides of an image | whose two s ides wy and xz are lying on the sides of $R_{n,m}$. A similar case will arise if we consider a line wx with w = (i,m) and x = (i,-m) for fixed i; -n<i<n.

Theorem 2.12: $\beta_{wt}(R_{n,m})=4$.

Proof: It was shown in [16] that the endpoints of one side of a rectangle form a metric basis. Further, it was shown that the endpoints of a diagonal do not resolve the vertices in $R_{n,m}$. So a WTR-set has at least three points in which two points are the endpoints of one side of $R_{n,m}$. Moreover, if the third point isany point of $R_{n,m}$, then not all the vertices are having different distances from two of these three vertices. Hence by Lemma 2.1, these three points do not form a WTR-set. This yields that $\beta_{wt}(R_{n,m})\ge 4$. Also, by Lemma 2.11, $\beta_{wt}(R_{n,m})\le 4$. It completes the proof.

Possibly to gain insight into the metric dimension, Chartrand *et al*. introduced the notion of a resolving partition and partition dimension. To define the partition dimension, the distance d(v,S) between a vertex v of G and $S\subseteq V(G)$ is defined as min d(v,s). Let Π = $\{S_1, S_2,\ldots, S_k\}$ be an ordered k-partition of V(G) and let v be a vertex of G, then the k-vector $(d(v,S_1),d(v,S_2),\ldots,d(v,S_k))$ is called the code, $c_\Pi(v)$, of v with respect to the partition Π. A partition Π is called a resolving partition if for distinct vertices u and v of G, $c_\Pi(u)\ne c_\Pi(v)$. The partition dimension of G is the cardinality of a minimum resolving partition, denoted by pd(G) [4].

Based on the Chartrand *et al*. method of vertex-partitioning, Javaid *et al* partition the vertex set of a connected graph G into classes in such a way that any two distinct vertices in G have different distances from at least two classes of the partition. They referred this partition as a fault-tolerent resolving partition of V(G), denoted by P(G) [12].

Now we show that every pair a,b of positive integers with 5≤a<b is realizable as the weak total metric and partition dimension of some connected graphs. Following observation will be useful in the proof of next theorem.

Observation 2.13

(i) Let Π be a fault-tolerant resolving partition of V(G) and $u,v\in V(G)$. If $d(u,w)=d(v,w)$ for all vertices $w\in V(G)\backslash\{u,v\}$, then u and v belong to different classes of Π.

(ii) If S is a set of k≥2 vertices in a connected graph G such that $d(u,x)=d(v,x)$ for all $u,v\in S$ and $x\in V(G)-\{u,v\}$, then every WTR-set must contain all the k vertices of S.

Theorem 2.14: For every pair (a,b) of positive integers with 5≤a<b, there exists a connected graph G such that P(G) = a and $\beta_{wt}(G)=b$.

Proof: Let H: $v_1v_2\ldots v_{4+b-a}$ be a path on 4+b-a vertices. We have the following two cases:

Case 1: (For even 4+b-a) Attach 'a' pendant vertices to one end point v_1 (say) and two pendant vertices to each vertex $v_2, v_3,\ldots,v_{\frac{1}{2}(4+b-a)}$ of H. Call this caterpillar G. Let us call 'a' pendant vertices p_i; 1≤i≤a (attached with endpoint v_1), two pendant vertices x_i, y_i; $1\le i\le\frac{1}{2}(2+b-a)$ (attached with $v_2, v_3,\ldots,v_{\frac{1}{2}(4+b-a)}$). Make a path H_1: $p_1p_2\ldots p_a$ and join each x_i with y_i; $1\le i\le\frac{1}{2}(2+b-a)$. Since 'a' vertices p_i's are attached with v_1, so, by Observation 2.13, P(G)≥a.

Now put these 'a' vertices into sets S_i; 1≤i≤a. Further, put v_1 into S_1, x_i; $1\le i\le\frac{1}{2}(2+b-a)$ into S_2, y_i; $1\le i\le\frac{1}{2}(2+b-a)$ into S_3 and v_i; $2\le i\le 3+b-a$ into S_4. Lastly, put v_{4+b-a} into S_5. Thus, we have the following a-partition $\Pi=\{S_1,S_2,\ldots S_a\}$ with

$$S_1 = \{p_1, v_1\}, S_2 = \{p_2, x_i : 1 \le i \le \tfrac{1}{2}(2+b-a)\}, S_3 = \{p_3, y_i : 1 \le i \le \tfrac{1}{2}(2+b-a)\}$$
$$S_4 = \{p_4, v_i : 2 \le i \le 3+b-a\}, S_5 = \{p_5, v_{4+b-a}\}$$

znd $p_j \in S_j$; $6 \le j \le a$. This partition Π of cardinality 'a' is a fault-tolerant resolving partition since every pair of vertices in G is resolved by at least two classes of Π. Thus, we conclude that P(G) = a.

There is one pendent path $H_1: p_1p_2\ldots p_a$ in G-v_1, one pendent path of order 2 in G-v_i where $2 \le i \le \frac{1}{2}(2+b-a)$ and two pedant paths, one of order 2 and one of order $\frac{1}{2}(2+b-a)$ in $G - v_{\frac{1}{2}(4+b-a)}$. Again, by using Observation 2.13, one can see that $\beta_{wt}(G) \ge b$. Take the set W defined as

$$W = \{p_1, p_3, p_4, \ldots, p_{a-2}, p_a, x_j, y_j;\ 1 \le j \le \tfrac{1}{2}(2+b-a)\}$$

Then this set is a minimum WTR-set for G of cardinality b. Therefore, $\beta_{wt}(G) = b$.

Case 2: (For odd 4+b-a) Attach 'a' pendant vertices to one end point v_1 (say), a single pendant vertex to $v_{\frac{1}{2}(5+b-a)}$ and two pendant vertices to each vertex $v_2, v_3, \ldots, v_{\frac{1}{2}(3+b-a)}$ of H. Call this caterpillar G.

Let us call 'a' pendant vertices p_i; $1 \le i \le a$ (attached with endpoint v_1), two pendant vertices x_i, y_i; $1 \le i \le \frac{1}{2}(1+b-a)$ (attached with $v_2, v_3, \ldots, v_{\frac{1}{2}(3+b-a)}$) and a single pendant vertex y (attached with $v_{\frac{1}{2}(5+b-a)}$). Make a path $H_1: p_1p_2\ldots p_a$ and join each x_i with y_i; $1 \le i \le \frac{1}{2}(1+b-a)$. On the same pattern as in Case 1, we can make the following minimum fault-tolerant resolving partition $\Pi = \{S_1, S_2, \ldots S_a\}$ with

$$S_1 = \{p_1, v_1\}, S_2 = \{p_2, x_i : 1 \le i \le \tfrac{1}{2}(1+b-a)\}, S_3 = \{p_3, y, y_i : 1 \le i \le \tfrac{1}{2}(1+b-a)\}$$
$$S_4 = \{p_4, v_i : 2 \le i \le 3+b-a\}, S_5 = \{p_5, v_{4+b-a}\}$$

and $p_j \in S_j$; $6 \le j \le a$ and the following is a minimum WTR-set

$$W = \{p_1, p_3, p_4, \ldots, p_{a-2}, p_a, x_j, y_j, y;\ 1 \le j \le \tfrac{1}{2}(1+b-a)\}$$

of cardinality b. Together with this and by using Observations 2.13, one can see that P(G) = a and $\beta_{wt}(G) = b$.

APPROXIMATING THE MAXIMUM ORDER OF A GRAPH

The following theorem approximate the maximum order of a connected graph G in terms of diameter and weak total metric dimension of G.

Theorem 3.1: Let G be a connected graph with weak total metric dimension β_{wt} and diameter D. Then

$$|V(G)| \le D^{\beta_{wt}-2}(D^2-1) - D^2 + \beta_{wt} + 1$$

Proof: Let $W = \{v_1, v_2, \ldots, v_{\beta_{wt}}\}$ be a WTMB of G and $U = V(G) \setminus W$. We will find the maximum cardinality of U such that the codes of any two vertices of U differ by at least one coordinate, also the codes of vertices of U and W differ by at least two coordinates. Note that, the only vertex v_i has ith coordinate 0 in its code, each other coordinate is an integer between 1 and D. Since weak total metric dimension is β_{wt} and diameter is D, we have $\beta_{wt}D^{\beta_{wt}-1}$ possible codes (called the basis codes) for the elements in WTMB and $D^{\beta_{wt}}$ possible codes for the elements in U.

Out of $\beta_{wt}D^{\beta_{wt}-1}$ possible basis codes, $D^{\beta_{wt}-1}$ have 0 at first coordinate, $D^{\beta_{wt}-1}$ have 0 at second coordinate and so on and the last $D^{\beta_{wt}-1}$ have 0 at β_{wt}th coordinate. From each of these $D^{\beta_{wt}-1}$ possible basis code, we can choose at most one code. Thus, we have β_{wt} such codes out of $\beta_{wt}D^{\beta_{wt}-1}$.

Place $D^{\beta_{wt}}$ possible codes for the elements in U column-wise in D columns as: place all those codes in ith column whose first coordinate is i where $1 \le i \le D$. Thus, we have exactly $D^{\beta_{wt}-1}$ codes in each column. In one of the columns, make segments of $D^{\beta_{wt}-1}$ codes with D codes in each segment, then one segment of D codes and exactly one code from each of the remaining segments do not differ by at least two coordinates from the codes of basis vertices, thus we are left with $(D^{\beta_{wt}-2} - D)(D-1)$ codes from this column and $D^{\beta_{wt}-1} - 1$ codes from each of the remaining D-1 columns which differ by at least one coordinate within the vertices of U and differ by at least two coordinates from the codes of basis vertices of W. Hence

$$|U| \le (D^{\beta_{wt}-2} - D)(D-1) + (D^{\beta_{wt}-1} - 1)(D-1)$$

Since $V(G) = W \cup U$, so the maximum order of G is at most $D^{\beta_{wt}-2}(D^2-1) - D^2 + \beta_{wt} + 1$.

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