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Examples in Cone Metric Spaces: A Survey

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Abstract: In this survey, we review many examples on cone metric spaces to verify some properties of cones on real Banach spaces and in the sequel, we shall present other examples in cone metric spaces that some properties are incorrect in these spaces but hold in ordinary case like as comparison test.

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INTRODUCTION

Let E be a real Banach space. A nonempty convex closed subset $P \subseteq E$ is called a cone in E if it satisfies:

i. P is closed, nonempty and $P \neq \{0\}$.
ii. $a,b \in R$, $a,b \geq 0$ and $x,y \in P$ imply that $ax+by \in P$.
iii. $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subseteq E$ that is, $x \leq y$ if and only if $y-x \in P$ Also we write $x << y$ if $y-x \in \text{int } P$ where int P denotes the interior of P

In the following we always suppose that E is a real Banach space, P is a cone in E and $\leq$ is partial ordering with respect to P

Definition 1.1: [1] The cone P is called

1. Normal if there exists a constant $K>0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.
2. Minihedral if $\sup(x,y)$ exists for all $x,y \in E$
3. Strongly minihedral if every subset of E which is bounded from above has a supremum.
4. Solid if int $P \neq \phi$
5. Generating if $E = P-P$.
6. Regular if every increasing sequence which is bounded from above is convergent. That is, if (x_n) is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq y$$

for some $y \in E$ then there is $x \in E$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

Definition 1.2: [2] Let X be a nonempty set. Assume that the mapping $d: X \times X \to E$ satisfies

i. $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only $x = y$.
ii. $d(x,y) = d(y,x)$ for all $x,y \in X$.
iii. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 1.3: Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

i. $\{x_n\}$ is said to be convergent to $x \in X$ whenever for every $c \in E$ with $0 << c$ there is N such that for all $n \geq N$, $d(x_n,x) << c$ that is, $\lim_{n\to\infty} x_n = x$.
ii. $\{x_n\}$ is called a Cauchy sequence in X whenever for every $c \in E$ with $0 << c$ there is N such that for all $m,n \geq N$ $d(x_n,x_m) << c$.
iii. (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

The following example states that sandwich theorem does not hold.

Example 1.4: [3] Let $E = C_R^1[0,1]$ with

$$\|x\| = \|x\|_\infty + \|x'\|_\infty$$

$$P = \{x \in E \mid x(t) \geq 0\}$$

This cone is non-normal. Consider,

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$$x_n(t) = \frac{t^n}{n} \text{ and } y_n(t) = \frac{1}{n}$$

Then $0 \le x_n \le y_n$ and $\lim_{n\to\infty} y_n = 0$. but

$$||x_n|| = \max\{t \in [0,1]: \frac{t^n}{n}\} + \max\{t \in [0,1]: t^{n-1}\}$$

So

$$||x_n|| = \frac{1}{n} + 1 > 1$$

hence x_n does not converge to zero. This shows us the sandwich theorem does not hold.

Example 1.5: [4] Let $E = C^1_R[0,1]$ with

$$||f|| = ||f||_\infty + ||f'||_\infty$$

$$P = \{f \in E \mid f(t) \ge 0\}$$

Put

$$x_n(t) = \frac{1-\sin nt}{n+2} \text{ and } y_n(t) = \frac{1+\sin nt}{n+2}$$

Then

$$0 \le x_n \le x_n + y_n,\ ||x_n|| = ||y_n|| = 1$$

and

$$x_n + y_n = \frac{2}{n+2} \to 0$$

The next example states cone can be non-normal.

Example 1.6: [5] Let $E = C^2_R[0,1]$ with

$$||f|| = ||f||_\infty + ||f'||_\infty$$

$$P = \{f \in E \mid f(t) \ge 0\}$$

For each $k \ge 1$ put $f(x) = x$ and $g(x) = x^{2k}$. Then $0 \le g \le f$, $||f|| = 2$ and $||g|| = 2k+1$ Since $||f|| < ||g||$, k is not normal constant of P. Therefore, P is not non-normal cone.

The next is normal cone with K>1.

Example 1.7: [5] Let with K>1. be given. Consider the real vector space with

$$E = \{ax + b : a, b \in R; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \ge 0, b \le 0\}$$

in E. The cone P is regular and so normal.

The following examples verify some properties of definition 1.1.

Example 1.8: [7] Let $E = R^n$ with

$$P = \{(x_1, \cdots, x_n) : x_i \ge 0, \forall i = 1, \cdots, n\}$$

The cone P is normal, generating, minihedral, strongly minihedral and solid.

Example 1.9: [7] Let $D \subseteq R^n$ be a compact set, E = C(D) and

$$P = \{f \in E \mid f(t) \ge 0, \forall x \in D\}$$

The cone P is normal, solid, generating and minihedral but is not strongly minihedral, P isn't regular.

Example 1.10: [7] Let (X, S, μ) be a finite measure space, S countably generated, $E = L^p(X)$, $l < p < \infty$ and

$$P = \{f \in E \mid f(t) \ge 0, [\mu] \text{a.e.on } X\}$$

The cone P is normal, generating, regular, minihedral and strongly minihedral and it isn't solid.

Example 1.11: Let $E = C^2_{R^+}[0,1]$ with norm

$$||f|| = ||f||_\infty + ||f'||_\infty$$

and

$$P = \{f \in E \mid f(t) \ge 0\}$$

that isn't normal cone by [5] and not minihedral by [1].

Example 1.12: [8] Let $E = R^2$ and

$$P = \{(x, y) : x, y \ge 0\}$$

The cone P is strongly minihedral in which each subset of P has infimum.

Example 1.13: Let $E = R^2$ and

$$P = \{(x, 0) : x \ge 0\}$$

This P is strongly minihedral but not minihedral by [1].

Example 1.14: Let $E = C_R[0,1]$ with the supremum norm and

$$P = \{f \in E \mid f(t) \ge 0\}$$

Then P is a cone with normal constant of K = 1.

The next examples are some of cone metrics.

Example 1.15: [2] Let $E = R^2$

$$P = \{(x,y): x, y \geq 0\}$$

X = R and d: X×X→E such that

$$d(x,y) = (|x-y|, \alpha|x-y|)$$

where α≥0 is a constant. Then (X, d) is a cone metric space.

Example 1.16: Let $E = R^n$ with

$$P = \{(x_1,\cdots,x_n) : x_i \geq 0, \forall i = 1,\cdots,n\}$$

X = R and d: X×X→E such that

$$d(x,y) = (|x-y|, \alpha_1|x-y|,\cdots,\alpha_{n-1}|x-y|)$$

where $\alpha_i \geq 0$ for all $1 \leq i \leq n-1$. Then (X, d) is a cone metric space.

Example 1.17: [6] Let

$$E = (C_R[0,\infty), ||.||_\infty)$$

$$P = \{f \in E \,|\, f(t) \geq 0\}$$

(X, ρ) a metric space and d: X×X→E defined by

$$d(x,y) = \rho(x,y)\varphi$$

where φ: $[0,1] \to R^+$ is continuous. Then (X, d) is a normal cone metric space and the normal constant of P is equal to K = 1.

Example 1.18: [6] Let q>0, $E = l^q$

$$P = \{\{x_n\} : x_n \geq 0, \forall n\}$$

(X, ρ) a metric space and defined by

$$d(x,y) = \{(\frac{\rho(x,y)}{2^n})^{1/q}\}_{n \geq 1}$$

Then (X, d) is a cone metric space and the normal constant of P is equal to K = 1.

Example 1.19: [4] Let E = $C_R[0,1]$ with the supremum norm and

$$P = \{f \in E \,|\, f(t) \geq 0\}$$

Then P is a cone with normal constant of K = 1. Define

$$d: X \times X \to E \text{ by } d(x,y) = |x-y|\varphi$$

where φ: $[0,1] \to R^+$ such that $\varphi(t) = e^t$. It is easy to see that d is a cone metric on X.

Example 1.20: [9] Let

$$E = (C_R[0,\infty), ||.||_\infty)$$

$$P = \{f \in E \,|\, f(t) \geq 0\}$$

(X, ρ) a metric space and d: X×X→E defined by d(x,y) = $f_{x,y}$ where $f_{x,y}(t) = |x-y|t$. Then (X, d) is a normal cone metric space and the normal constant of P is equal to K = 1.

Example 1.21: [16] Let $M = E = C^2_{R^+}[0,1]$ with norm

$$||f|| = ||f||_\infty + ||f'||_\infty$$

and

$$P = \{f \in E \,|\, f(t) \geq 0\}$$

that isn't normal cone by [5]. Consider

$$x_n(t) = \frac{1-\sin nt}{n+2} \text{ and } y_n(t) = \frac{1+\sin nt}{n+2}$$

so

$$0 \leq x_n \leq x_n + y_n,\ ||x_n|| = ||y_n|| = 1$$

and

$$x_n + y_n = \frac{2}{n+2} \to 0\ [1]$$

Define cone metric d: M×M→E with

$$d(f,g) = f + g, \text{ for } f = g;\ d(f,f) = 0.$$

Since $x_n \ll c$ namely $d(x_n, 0) \ll c$ but $d(x_n,0)$ doesn't tend to zero. Indeed $x_n \to 0$ in (M, d) but x_n again does not tend to zero in E. Even for n>m

$$d(x_n,x_m) = ||x_n + x_m|| \ll c$$

and

$$d(x_n,x_m) = ||x_n + x_m|| = 2$$

in particular $d(x_n, x_{n+1}) \ll c$ but $d(x_n, x_{n+1})$ does not tend to zero.

MAIN RESULTS

In this section we obtain other examples in cone metric spaces that some properties are incorrect in these spaces but hold in ordinary case. For instance, two examples are presented: first one states comparison test does not hold and the second example is for normal cone which we can find two members of cone that $f \geq g$ but $\|f\| > \|g\|$.

Example 2.1: Let $E = C^1_R[0,1]$ with norm

$$\|x\| = \|x\|_\infty + \|x'\|_\infty$$

and

$$P = \{x \in E \mid x(t) \geq 0\}$$

that isn't normal cone.
For all $n \geq 1$ and $t \in [0,1]$ put

$$x_n(t) = \frac{t^{(n-1)^2}}{(n-1)^2+1} - \frac{t^{n^2}}{n^2+1} \text{ and } y_n(t) = \frac{2}{n^2}$$

So $0 \leq x_n \leq y_n$, and

$$s_n(t) = \sum_{k=1}^{n} x_k(t) = 1 - \frac{t^{n^2}}{n^2+1}$$

Therefore

$$\begin{aligned} \|s_n - s_m\| &= \|s_n - s_m\|_\infty + \|(s_n - s_m)'\|_\infty \\ &= \left\|\frac{t^{m^2}}{m^2+1} - \frac{t^{n^2}}{n^2+1}\right\|_\infty + \left\|\frac{m^2 t^{m^2-1}}{m^2+1} - \frac{n^2 t^{n^2-1}}{n^2+1}\right\|_\infty \\ &= \frac{1}{m^2+1} + \frac{m^2}{m^2+1} = 1 \end{aligned}$$

for all m,n So $\{s_n\}$ is not Cauchy sequence, namely $\sum_{k=1}^{\infty} x_k(t)$ is divergent, but

$$\sum_{k=1}^{\infty} y_k(t) = \sum_{k=1}^{\infty} \frac{2}{k^2}$$

is convergent. This means comparison test does not hold for series.

Example 2.2: Let E be a real vector space

$$E = \{ax + b : a, b \in R; x \in [\tfrac{1}{2}, 1]\}$$

with supremum norm and

$$P = \{ax + b : a \leq 0, b \geq 0\}$$

So P is a normal cone in E with constant K>1. Define

$$f(x) = -2x + 10, g(x) = -6x + 11 \in P$$

Then $f \leq g$ as $g(x) - f(x) = -3x + 1 \in P$. But

$$\|f\| = f(\tfrac{1}{2}) = 9, \|g\| = g(\tfrac{1}{2}) = 8$$

Therefore, $f \leq g$ but $\|f\| > \|g\|$.

CONCLUSION

According to example 1.4 we obtained that sandwich theorem didn't hold in cone metric space. We provided also example 2.1 that stated comparison test didn't hold in such space. And in the example 2.2 we saw that we may find two elements such that as $f \leq g$ but $\|f\| > \|g\|$.

These examples led us to find other examples or properties which may be held in ordinary spaces but don't hold in cone metric spaces.

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REFERENCES

1. Deimling, K., 1985. Nonlinear Functional Analysis, Springer-Verlage.
2. Long-Guang, H. and Z. Xian, 2007. Cone Metric Spaces and Fixed Point Theorems of Contractive Mapping. J. Math. Anal. Appl., 322: 1468-1476.
3. Slobodanka Jankovic, Zoran Kadelburg and Stojan Radenovic, 2011. On cone metric spaces: A survey, Nonlinear Analysis: Theory, Methods and Applications, 74 (7): 2591-2601.
4. Kadelburg, Z., S. Radenovic and V. Rakocevic, 2009. Remarks on "Qusicontraction on a cone metric space". Applid Mathematics Letters, In Press.
5. Rezapour, Sh. and R. Hamlbarani, 2008. Some Notes on the Paper Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings. J. Math. Anal. Appl., 345: 719-724.
6. Haghi, R.H. and Sh. Rezapour, 2010. Fixed points of multifunctions on regular cone metric spaces. Expositiones Mathematicae, 28 (1): 71-77.

7. Denkoweski, Z., S. Migorski and N.S. Apagergiou, 2003. An Introduction to Nonlinear Analysis: applications. Kluwer Academic/Plenum Publishers, New York.
8. Turkoglu, D., M. Abuloha and T. Abdeljawad, 2009. KKM mappings in cone metric spaces and some fixed point theorems. To appear in: Nonlinear Analysis.
9. Rezapour, Sh., 2010. Fixed points of multifunctions on regular cone metric spaces. Expositions Mathematicae, 28 (1): 71-77.
10. Abbas, M. and G. Jungck, 2008. Common Fixed Point Results for Noncommuting Mapping Without Continuity in Cone Metric Spaces. J. Math. Anal. and Appl., 341 (1): 416-420.
11. Abbas, M. and B.E. Rohades, 2009. Fixed Point and periodic Point Results in Cone Metric Spaces. Applied Mathematics Letters, 22: 511-515.
12. Illic, D. and V. Rakocevic, 2008. Common Fixed Point for Maps on Cone Metric Spaces. J. Math. Anal. and Appl., 341 (2): 876-882.
13. Illic, D. and V. Rakocveic, 2009. Quasi-contraction on Cone Metric Space. Applied Mathematics Letters, 22: 728-731.
14. Raja, P. and S.M. Vaezpour, 2008. Some Extentions of Banach's Contarction Principle in Complete Cone Metric Spaces. Fixed Point Theory and Application, Volume 2008, Article ID 768294, pp: 11.
15. Wardowski, D., 2009. Endpoints and Fixed Points of set-valued Contractions in Cone Metric Spaces. Nonliner Analysis: Theory. Methods and Applications, 71: 512-516.
16. Asadi, M.H. Soleimani and S.M. Vaezpour, 2009. An Order on Subsets of Cone Metric Spaces and Fixed Points of Set-Valued Contractions. Fixed Point Theory and Applications, 2009: Article ID 723203, pp: 8, doi:10.1155/2009/723203.