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Chen and Stenlund Relations for Stirling Numbers

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Abstract: We study the expressions of Stenlund and Chen involving Stirling numbers.

Key words: Stirling numbers of first and second kind - Olson's identity

INTRODUCTION

Stenlund [1] obtained the relations:

$$A \equiv \sum_{j=1}^{m} S_m^{(j)} \sum_{k=1}^{j} S_j^{[k]} = 1, \quad B \equiv \sum_{j=1}^{m} S_m^{[j]} \sum_{k=1}^{j} S_j^{(k)} = 1, \tag{1}$$

Involving the Stirling numbers [2], which are obvious because:

$$A = \sum_{k=1}^{m} \sum_{j=k}^{m} S_m^{(j)} S_j^{[k]} = \sum_{k=1}^{m} \delta_{km} = 1, \; B = \sum_{k=1}^{m} \sum_{j=k}^{m} S_m^{[j]} \; S_j^{(k)} = \sum_{k=1}^{m} \delta_{km} = 1.$$

In [3] were deduced the expressions:

$$C \equiv \sum_{j=1}^{m-1} S_m^{(j)} \; S_j^{[1]} = -S_m^{[1]} \, , m \geq 2, \; D \equiv \sum_{m=1}^{j-1} S_j^{[m]} \, S_m^{(1)} = -S_j^{(1)} \, , \; j \geq 2, \tag{2}$$

Which are evident because:

$$C = \sum_{j=1}^{m} S_m^{(j)} \; S_j^{[1]} - S_m^{[1]} = \delta_{1m} - S_m^{[1]} = -S_m^{[1]} \, , \; D = \sum_{m=1}^{j} S_j^{[m]} \; S_m^{(1)} - S_j^{(1)} = \delta_{1j} - S_j^{(1)} = -S_j^{(1)} \, .$$

Chen [4, 5] showed the following convolution formula:

$$\sum_{r=0}^{n} \sum_{j=k-r}^{m} r^{m-j} \binom{m}{j} S_j^{[k-r]} \, S_j^{[r]} = S_{m+n}^{[k]} \, , \; 0 \leq k \leq m+n, \; m,n \geq 0, \tag{3}$$

Whose inversion gives the relation:

$$\sum_{r=0}^{n} S_{m+r}^{[k]} \; S_n^{(r)} = \sum_{j=k-n}^{m} n^{m-j} \binom{m}{j} S_j^{[k-n]} \, , \; m,n \geq 0, \; 0 \leq k \leq m+n. \tag{4}$$

From (4) with $k = 1$ results the property [2]:

$$\sum_{r=0}^{n} S_n^{(r)} = 0, \; n \geq 2 \, , \tag{5}$$

and if we employ $k = n + 1$ into (4):

$$\sum_{r=0}^{n} S_{m+r}^{[n+1]} \, S_n^{(r)} = (1+n)^m - n^m, \; m \geq 1, \tag{6}$$

which for $n = 1, 2$ implies the values [2]:

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$$S_{m+1}^{[2]} = 2^m - 1\,,\ S_{m+2}^{[3]} - S_{m+1}^{[3]} = 3^m - 2^m\,,\ S_{m+1}^{[3]} = \frac{1}{2}(3^m + 1 - 2^{m+1}),\ m \geq 1. \tag{7}$$

From (4) with $m = n - 1 \geq 0$ is immediate the identity:

$$\sum_{r=0}^{n} S_{n+r-1}^{[k]}\ S_n^{(r)} = \begin{cases} 0\,,\ k < n \\ n^{n-1}, k = n \end{cases},\ n \geq 1\,, \tag{8}$$

Which is a particular case of the Olson's relation [2, 6]:

$$\sum_{r=0}^{n} S_{n+r-1}^{[k]}\ S_q^{(r)} = \begin{cases} 0\,,\ k < q \\ q^{n-1}, k = q \end{cases},\ n \geq 1\,. \tag{9}$$

Besides, from (4) for $m = n - 1 \geq 0, k > n$:

$$\sum_{r=0}^{n} S_{n+r-1}^{[k]} S_n^{(r)} = \sum_{j=k-n}^{n-1} n^{n-1-j} \binom{n-1}{j} S_j^{[k-n]}\,,\ 1 \leq n < k \leq 2n - 1. \tag{10}$$

It is easy to see that (4) allows extend (8):

$$\sum_{r=0}^{n} S_{m+r}^{[k]}\ S_n^{(r)} = \begin{cases} 0\,,\ k < n \\ n^m, k = n \end{cases},\ n \geq 1\,, m \geq 0. \tag{11}$$

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