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On the Roman's Identity Involving Bernoulli and Stirling Numbers

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Abstract: We give a proof of the Roman's formula involving Bernoulli and Stirling numbers.

Key words: Stirling numbers - Roman's identity - Polynomials and numbers of Bernoulli

INTRODUCTION

Roman [1, 2] obtained the identity:

$$S_{n-1}^{[k-1]} = \frac{1}{n\,(k-1)!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k+j}\, B_n(j), \quad 1 \leq k \leq n, \tag{1}$$

where $B_n(x)$ and $S_r^{[m]}$ are the Bernoulli polynomials [3-7] and the Stirling numbers of the second kind [8-11], respectively:

$$B_n(j) = \sum_{r=0}^{n} \binom{n}{r} B_r\, j^{n-r}, \tag{2}$$

$$S_m^{[k]} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^m, \tag{3}$$

With $B_r \equiv B_r(0)$ the Bernoulli numbers [4, 5, 8-10], and is very known the relationship [12]:

$$B_r = \sum_{k=0}^{r} \frac{(-1)^k\, k!}{k+1}\, S_r^{[k]}, \tag{4}$$

and the convolution form [9]:

$$\sum_{r=0}^{n} \binom{n}{r} S_r^{[q]}\; S_{n-r}^{[k]} = \binom{q+k}{q} S_n^{[q+k]}. \tag{5}$$

We have the relation [9]:

$$\sum_{l=k}^{n} \frac{1}{l}\, S_{n-1}^{[l-1]}\, S_l^{(k)} = \binom{n}{k} \frac{B_{n-k}}{n}, \quad 1 \leq k \leq n, \tag{6}$$

Whose inversion implies the property:

$$\sum_{r=0}^{n} \binom{n}{r} B_r\, S_{n-r}^{[k]} = \frac{n}{k}\, S_{n-1}^{[k-1]}, \quad n \geq k \geq 1, \tag{7}$$

Therefore, from (7):

$$S_{n-1}^{[k-1]} \overset{(3)}{=} \frac{(-1)^k}{(k-1)!\,n} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{r=0}^{n} \binom{n}{r} B_r\, j^{n-r} \overset{(2)}{=} \frac{1}{(k-1)!\,n} \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} B_n(j),$$

In harmony with (1), q.e.d.

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The application of (4) into (7) gives:

$$S_{n-1}^{[k-1]} = \frac{k}{n} \sum_{q=0}^{n} \frac{(-1)^q\, q!}{q+1} \sum_{r=0}^{n} \binom{n}{r} S_r^{[q]}\, S_{n-r}^{[k]} \overset{(5)}{=} \frac{1}{n\,(k-1)!} \sum_{q=0}^{n-k} \frac{(-1)^q\,(q+k)!}{q+1}\, S_n^{[q+k]}\,,$$

That is:

$$S_{n-1}^{[k-1]} = \frac{(-1)^k}{n\,(k-1)!} \sum_{j=k}^{n} \frac{(-1)^j\, j!}{j-k+1}\, S_n^{[j]}\,, \quad 1 \le k \le n\,. \tag{8}$$

whose inversion allows deduce the identity:

$$\sum_{l=k}^{n-1} (l+1)\, S_n^{(l+1)}\, S_l^{[k]} = \frac{(-1)^{n-k+1}\, n!}{k!\,(n-k)}\,, \quad 0 \le k \le n-1, \tag{9}$$

and the inversion of (9) implies the expression:

$$\sum_{l=k}^{n-1} (-1)^l\; l!\,(n-l)\, S_l^{(k)} = \frac{(-1)^{n+1}\,(k+1)}{n!}\, S_n^{(k+1)}\,, \quad n-1 \ge k \ge 0, \tag{10}$$

where, for example, we can employ the values $k = 0, 1$ to deduce the known relations [9]:

$$S_n^{(1)} = (-1)^{n+1}\,(n-1)!\,, \quad S_n^{(2)} = (-1)^n\,(n-1)!\; H_{n-1}\,, \tag{11}$$

Involving harmonic numbers [9, 13].

The application of (1) into (4) gives the interesting property:

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} \binom{n}{j} B_n(j) = H_n\, B_n + n\, B_{n-1}\,, \quad n \ge 1. \tag{12}$$

Besides, (7) and (8) permit obtain the following identities:

$$\sum_{r=0}^{n-k} \frac{(-1)^r\,(r+k)!\,(n-k+n\,r)}{r+1}\, S_{n+1}^{[r+k+1]} = 0, \quad 1 \le k \le n\,,$$

$$\sum_{k=0}^{n} (-1)^k\; k!\; S_{n+1}^{[k+1]} = \delta_{n0}\,, \quad \sum_{k=1}^{n-1} (-1)^k\,(k-1)!\; S_n^{[k+1]} = 1-n\,, \tag{13}$$

$$\sum_{k=0}^{n-1} (-1)^k\, k!\,(k+2)\, S_{n+1}^{[k+2]} = 1+n\,.$$

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