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On a Recent Formula for Bernoulli Numbers Involving Stirling Numbers

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Abstract: Recently, Jha obtained an interesting relation for Bernoulli numbers in terms of Stirling numbers of the second kind. Here we exhibit an alternative deduction of this formula.

Key words: Stirling numbers, Fukuhara-Kawazumi-Kuno's relation, Bernoulli numbers.

INTRODUCTION

Jha [1] obtained the following expression for Bernoulli numbers [2]:

$$B_{m+n} = \sum_{k=0}^{n} \sum_{r=0}^{m} \frac{(-1)^{k+r}\,(k!\,r!)^2}{(k+r+1)!}\, S_m^{[r]}\, S_n^{[k]}\,, \qquad m,n \geq 0, \tag{1}$$

Involving the Stirling numbers of the second kind [3]. For $m = n = 0$ & $m = 0$, with n arbitrary, (1) implies $B_0 = 1$ and [2-4]:

$$B_n = \sum_{k=0}^{n} \frac{(-1)^k\, k!}{k+1}\, S_n^{[k]}\,, \tag{2}$$

and for m = 1:

$$B_{n+1} = \sum_{k=0}^{n} \frac{(-1)^{k-1}\, k!}{(k+1)\,(k+2)}\, S_n^{[k]}\,, \tag{3}$$

Deduced by Jha [5]; thus, here we shall consider (1) for $m,n \geq 1$.

Fukuhara-Kawazumi-Kuno [6] showed the identity:

$$B_N = (-1)^M \sum_{j=1}^{Q+1} \frac{(-1)^{j+1}}{j} \binom{Q+1}{j} \sum_{q=1}^{j-1} q^M\,(j-q)^{N-M}, 0 \leq M \leq N \leq Q,\ N \geq 2, \tag{4}$$

which for $N = Q = m,\ M = 0$ gives the Kronecker's formula [7, 8]:

$$B_m = \sum_{k=2}^{m+1} \frac{(-1)^{k+1}}{k} \binom{m+1}{k} \sum_{r=1}^{k-1} r^m\,, \qquad m \geq 2. \tag{5}$$

Now we apply (4) for $M = 0,\ N = k+r,\ 0 \leq k+r \leq Q = m+n,\ \ m,n \geq 1$:

$$B_{k+r} = \sum_{j=1}^{m+n+1} \frac{(-1)^{j+1}}{j} \binom{m+n+1}{j} \sum_{q=1}^{j-1} (j-q)^{k+r}\,,$$

which can be multiplied by Stirling numbers of the first kind [2, 3] to obtain:

$$\sum_{k=0}^{n} \sum_{r=0}^{m} B_{k+r}\, S_n^{(k)}\, S_m^{(r)} =$$

$$= \sum_{q=1}^{m+n+1} \sum_{l=0}^{m+n+1-q} \frac{(-1)^{l+q+1}}{l+q} \binom{m+n+1}{l+q} \sum_{k=0}^{n} \sum_{r=0}^{m} l^{k+r}\ S_n^{(k)}\, S_m^{(r)},$$

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$$= m!\,n!\ \sum_{q=1}^{n+m+1} \sum_{l=0}^{m+n+1-q} \frac{(-1)^{l+q+1}}{l+q} \binom{m+n+1}{l+q} \binom{l}{m} \binom{l}{n},$$

$$= m!\,n!\ \sum_{q=1}^{m+n+1} \sum_{j=q}^{m+n+1} \frac{(-1)^{j+1}}{j} \binom{m+n+1}{j} \binom{j-q}{m} \binom{j-q}{n},$$

$$= m!\,n!\ \sum_{j=\max(m+1,n+1)}^{m+n+1} \frac{(-1)^{j+1}}{j} \binom{m+n+1}{j} \sum_{t=\max(m,n)}^{j-1} \binom{t}{m} \binom{t}{n}, \tag{6}$$

and this final sum can be evaluated with MAPLE [we acknowledge to Prof. Doron Zeilberger, Rutgers University, his help with evaluation of (6)] to deduce the result:

$$A(n,m) = A(m,n) \equiv \sum_{k=0}^{n} \sum_{r=0}^{m} B_{k+r}\ S_n^{(k)}\ S_m^{(r)} = \frac{(-1)^{m+n}\,(m!\,n!)^2}{(m+n+1)!}. \tag{7}$$

The relation (7) also was verified for many values of m and n using Wolfram Mathematica 12.

We have the inversion formula [3, 9]:

$$\sum_{k=0}^{n} f(k)\ S_n^{(k)} = g(n) \qquad \therefore \qquad \sum_{k=0}^{n} g(k)\ S_n^{[k]} = f(k), \tag{8}$$

whose application to (7) implies the identity:

$$\sum_{k=0}^{n} B_{k+m}\ S_n^{(k)} = (-1)^n (n!)^2 \sum_{r=0}^{m} \frac{(-1)^r\,(r!)^2}{(n+r+1)!}\ S_m^{[r]}. \tag{9}$$

Finally, the use of (8) in (9) gives the expression (1) obtained by Jha [1], q.e.d.

For example, in (9) we can employ $m = 2, 3$ to deduce the expressions:

$$\sum_{k=0}^{n} B_{k+2}\ S_n^{(k)} = \frac{(-1)^n\,(n!)^2\,(1-n)}{(n+3)!}, \qquad \sum_{k=0}^{n} B_{k+3}\ S_n^{(k)} = \frac{(-1)^n\,(n!)^2\,n\,(5-n)}{(n+4)!}, \tag{10}$$

That is:

$$B_{n+2} = \sum_{k=0}^{n} \frac{(-1)^k\,(k!)^2\,(1-k)}{(k+3)!}\ S_n^{[k]}, \qquad B_{n+3} = \sum_{k=0}^{n} \frac{(-1)^k\,(k!)^2\,k\,(5-k)}{(k+4)!}\ S_n^{[k]}. \tag{11}$$

Besides, from (7) it is possible construct the relations:

$$A(n,m+1) = -\frac{(m+1)^2}{n+m+2}\ A(n,m), \qquad A(n-1,m+1) = \left(\frac{m+1}{n}\right)^2 A(n,m),$$

$$A(n,m-1) = -\frac{n+m+1}{m^2}\ A(n,m), \qquad A(n+1,n+1) = \frac{(n+1)^3}{2(2n+3)}\ A(n,n),$$

$$A(n,n) = \frac{(n!)^4}{(2n+1)!},\ \ A(n,n+2) = \frac{(n+1)(n+2)^2}{2(2n+3)}\ A(n,n) = \left(\frac{n+2}{n+1}\right)^2 A(n+1,n+1), \tag{12}$$

$$\sum_{k=0}^{n} \sum_{r=1}^{n+1} B_{k+r}\ S_n^{(r-1)} S_n^{(k)} = \sum_{k=0}^{n} \sum_{j=0}^{n} B_{j+k+1}\, S_n^{(j)}\, S_n^{(k)},$$

$$= A(n,n+1) + n\,A(n,n) = \frac{n-1}{2}\ A(n,n),$$

$$\sum_{k=1}^{n+1} \sum_{r=1}^{n+1} B_{k+r}\ S_n^{(r-1)}\, S_n^{(k-1)} = \sum_{q=0}^{n} \sum_{j=0}^{n} B_{j+q+2}\ S_n^{(j)}\, S_n^{(q)},$$

$$= A(n+1,n+1) - n\,A(n,n) = \frac{n^3 - n^2 - 3n + 1}{2(2n+3)}\ A(n,n).$$

As a note, from (4) for $M = N = Q \geq 2$:

$$B_N = (-1)^N \sum_{j=2}^{N+1} \frac{(-1)^{j+1}}{j} \binom{N+1}{j} \sum_{q=1}^{j-1} q^N \overset{(5)}{=} (-1)^N B_N ,$$

Therefore $B_m = 0,\ m = 3, 5, 7, 9, \ldots$; and for $N = Q = M + 1$:

$$[1 + (-1)^M]\, B_{M+1} = \sum_{j=2}^{M+2} (-1)^{j+1} \binom{M+2}{j} \sum_{q=1}^{j-1} q^M = 0, \quad \forall\ M \geq 1. \tag{13}$$

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