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On the Bessel and Stirling Numbers

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Abstract: We study the identities obtained by Yang-Qiao and Mansour-Schork-Shattuck involving Stirling and Bessel numbers.

Key words: Bernoulli - Bessel and Stirling numbers

INTRODUCTION

The Bessel numbers of the first and second kind are given by [1-3]:

$$b_n^{(k)} = \frac{(-1)^{n-k}\,(2n-k-1)!}{2^{n-k}\,(k-1)!\,(n-k)!}, \qquad b_n^{[k]} = \frac{n!}{2^{n-k}\,(2k-n)!\,(n-k)!}, \tag{1}$$

Respectively, with the properties:

$$b_n^{[k]} = (-1)^{n-k}\, b_{k+1}^{(2k-n+1)}, \qquad \sum_{k=m}^{n} b_n^{[k]}\, b_k^{(m)} = \sum_{k=m}^{n} b_n^{(k)}\, b_k^{[m]} = \delta_{mn}. \tag{2}$$

Yang-Qiao [1, 3] obtained the relations:

$$\sum_{r=k}^{n} \frac{1}{2^r}\, S_r^{[k]}\, S_n^{(r)} = 2^{-n}\, b_n^{(k)} = \frac{(-1)^{n-k}\,(2n-k-1)!}{2^{2n-k}\,(k-1)!\,(n-k)!}, \tag{3}$$

$$\sum_{r=k}^{n} 2^r\, S_r^{[k]}\, S_n^{(r)} = 2^k\, b_n^{[k]} = \frac{n!}{2^{n-2k}\,(2k-n)!\,(n-k)!}, \tag{4}$$

Involving Stirling numbers of the first and second kind [4-6].

We know the inversion formula [5-8]:

$$\sum_{r=k}^{n} f(r)\, S_n^{(r)} = g(k) \qquad \therefore \qquad \sum_{r=k}^{n} g(r)\, S_n^{[r]} = f(k), \tag{5}$$

Whose application to (3) and (4) implies the identities:

$$S_n^{[k]} = \sum_{r=k}^{n} 2^{n-r}\, b_r^{(k)}\, S_n^{[r]} = \frac{(-1)^k\, 2^{n+k}}{(k-1)!} \sum_{r=k}^{n} \frac{(-1)^r\,(2r-k-1)!}{2^{2r}\,(r-k)!}\, S_n^{[r]}, \tag{6}$$

$$S_n^{[k]} = 2^{k-n} \sum_{r=k}^{n} b_r^{[k]}\, S_n^{[r]} = 2^{2k-n} \sum_{r=k}^{n} \frac{r!}{2^r\,(2k-r)!\,(r-k)!}\, S_n^{[r]}. \tag{7}$$

We may remember the following expression [5, 9-12] for the Bernoulli numbers [11, 13]:

$$\sum_{k=0}^{n} \frac{(-1)^k\, k!}{k+1}\, S_n^{[k]} = B_n, \tag{8}$$

where we can use (6), (7), and after (5) to deduce the relations:

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$$\sum_{r=0}^{n} \frac{1}{2^r} B_r S_n^{(r)} = \frac{1}{2^n} \sum_{k=0}^{n} \frac{(-1)^k k!}{k+1} b_n^{(k)}, \qquad \sum_{r=0}^{n} 2^r B_r S_n^{(r)} = \sum_{k=0}^{n} \frac{(-2)^k k!}{k+1} b_n^{[k]}. \tag{9}$$

We have the property [2, 3]:

$$\sum_{r=k}^{n} (-z)^r S_r^{[k]} S_n^{(r)} = (-z)^n G_{\frac{1}{z+1}, \frac{z+1}{z}}(n,k), \tag{10}$$

Involving the generalized Stirling numbers $G_{s,t}(n,k)$:

$$G_{1,-1}(n,k) = S_n^{(k)}, \qquad G_{0,1}(n,k) = S_n^{[k]}, \qquad G_{2,-1}(n,k) = b_n^{(k)}, \qquad G_{-1,1}(n,k) = b_n^{[k]}. \tag{11}$$

From (10) for $z = \frac{1}{2}$:

$$\sum_{r=k}^{n} \frac{(-1)^r}{2^r} S_r^{[k]} S_n^{(r)} = \frac{(-1)^n}{2^n} G_{\frac{2}{3},3}(n,k) = \frac{(-1)^{n-k} n!}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-1+\frac{j}{2}}{n}, \tag{12}$$

where it was employed the formula [3, 14]:

$$G_{\frac{q-1}{q},q}(n,k) = \frac{(q-1)^n n!}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \binom{n-1+\frac{r}{q-1}}{n}, \tag{13}$$

With $q = 3$. The application of (5) to (12) implies the identity:

$$S_n^{[k]} = \frac{(-1)^{n-k} 2^n}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{r=k}^{n} (-1)^r \frac{\Gamma(\frac{j}{2}+r)}{\Gamma(\frac{j}{2})} S_n^{[r]}. \tag{14}$$

Similarly, from (10) for $z = 2$:

$$\sum_{r=k}^{n} (-1)^r 2^r S_r^{[k]} S_n^{(r)} = (-1)^n 2^n G_{\frac{1}{3},\frac{3}{2}}(n,k) = (-1)^{n-k} \sum_{j=0}^{k} \frac{(-1)^j (n-1+2j)!}{j!\,(k-j)!\,(2j-1)!}, \tag{15}$$

where it was applied (13) with $q = \frac{3}{2}$. The inversion of (15) via (5) gives the expression:

$$S_n^{[k]} = \frac{(-1)^{n-k}}{2^n} \sum_{j=0}^{k} \frac{(-1)^j}{j!\,(k-j)!\,(2j-1)!} \sum_{r=k}^{n} (-1)^r (r-1+2j)!\, S_n^{[r]}. \tag{16}$$

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