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Some Applications of the Time Evolution Operator

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Abstract: We employ factorizations of the time evolution operator to obtain the propagator of free particle, harmonic oscillator and linear potential in quantum mechanics.

Key words: Time evolution operator · Propagator in quantum physics · Harmonic oscillator

INTRODUCTION

The propagator in quantum mechanics is given by [1]:

$$K(z_b,t_b,z_a t_a) = < z_a \mid U(t_b,t_a) \mid z_a >, \tag{1}$$

with the time evolution operator:

$$U(t_b,t_a) = \exp\left(-\frac{i}{\hbar}(t_b - t_a)H\right), \tag{2}$$

where H is the Hamiltonian operator for the system under analysis; thus:

$$\mid \psi(t_b) >= U(t_b,t_a) \mid \psi(t_a) >, \tag{3}$$

$$\psi(z_a,t_b) = < z_b \mid \psi(t_b) >= \int dz_a K(z_b t_b, z_a t_a) \psi(z_a,t_a). \tag{4}$$

In Sec. 2 we use (1) and (2) to determine the propagator for a free particle, harmonic oscillator and linear potential via certain factorizations of $U(t_b, t_a)$.

Propagators:

- Free particle.

$$K(z_b t_b, z_a t_b) = < z_b \exp(-\frac{i}{\hbar}(t_b - t_a)\frac{\hat{p}^2}{2m}) \mid z_a >,$$

$$= \int\int dp' \;\; dp" < z_b \mid p" >< p" \mid p' > \exp\left(-\frac{i}{\hbar}(t_b - t_a)\frac{p'^2}{2m}\right) < p' \mid z_a >,$$

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where we can employ the relations:

$$< z_b \mid p" >= \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p" z_b}, \quad < p" \mid p' >= \delta(p" - p'), \quad < p' z_a >= \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar} p' z_a}, \tag{5}$$

to obtain the expression:

$$K(z_b t_b, z_a t_a) = \frac{1}{2\pi\hbar} \int dp' \exp\left(-\frac{i}{\hbar}(t_b - t_a) \frac{p'^2}{2m} + \frac{i}{\hbar}(z_b - z_a) p' \right). \tag{6}$$

We know the property [2]:

$$\int_{-\infty}^{\infty} dx \exp(-i\alpha x^2 \pm i\beta x) = \sqrt{\frac{\pi}{i\alpha}} \exp\left(\frac{i\beta^2}{4\alpha} \right), \tag{7}$$

whose application in (6) implies the following propagator [1, 3, 4]:

$$K(z_b t_b, z_a t_a) = \sqrt{\frac{m}{2i\pi\hbar(t_b - t_a)}} \exp\left(\frac{im}{2\hbar} . \frac{(z_b - z_a)^2}{t_b - t_a} \right). \tag{8}$$

- Harmonic oscillator.

$$U(t_b, t_a) = \exp\left[-\frac{i}{\hbar}(t_b - t_a) \left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2} z^2 \right) \right] = \exp[c\left(q\frac{d^2}{dz^2} + \frac{z^2}{q} \right)], c = \frac{\omega}{2}(t_b - t_a), q = \frac{i\hbar}{m\omega}, \tag{9}$$

which accepts the decomposition [5-8]:

$$U(t_b, t_a) = \exp(\frac{z^2}{2q} \tan c) \exp\left(-\frac{q}{2\hbar^2} \sin(2c) \hat{p}^2 \right) \exp(\frac{z^2}{2q} \tan c), \ \hat{p} = -i\hbar \frac{d}{dz}, \tag{10}$$

then from (1), (5) and (10):

$$K(z_b t_b, z_a t_a) = \frac{1}{2\pi\hbar} \exp\left(\frac{\tan c}{2q}(z_a^2 + z_b^2) \right) \int dp' \exp\left[-\frac{i}{2m\omega\hbar} \sin(2c) p'^2 + \frac{i}{\hbar}(z_b - z_a) p' \right],$$

where we can use (7) to deduce the propagator [4, 9]:

$$K(z_b t_b, z_a t_a) = \sqrt{\frac{m\omega}{2\pi\hbar i \sin(2c)}} \exp\left[\frac{\tan c}{2q}(z_a^2 + z_b^2) + \frac{im\omega}{2\hbar \sin(2c)}(z_b - z_a)^2 \right], \tag{11}$$

$$= \sqrt{\frac{m\omega}{i2\pi\hbar i \sin(\omega(t_b - t_a))}} \exp\left\{ \frac{im\omega}{2\hbar \sin(\omega(t_b - t_a))} [z_a^2 + z_b^2) \cos(\omega(t_b - t_a)) - 2z_a z_b] \right\}.$$

- Linear potential.

$$U(t_b, t_a) = \exp[-\frac{i\Delta t}{\hbar} \left(\frac{\hat{p}^2}{2m} + kz \right)], \quad \Delta t = t_b - t_a, \tag{12}$$

which admits the splitting [8, 10]:

$$U(t_b,t_a)=\exp\left(-\frac{ik^2}{6m\hbar}(\Delta t)^3\right)\exp\left(-\frac{ik}{2m\hbar}(\Delta t)^2\,\hat{p}\right)\exp\left(-\frac{i}{2m\hbar}\Delta t.\hat{p}^2\right)\exp(-\frac{ik}{\hbar}\Delta t.z), \tag{13}$$

thus from (1), (5) and (13):

$$K(z_bt_b,z_at_a)=\exp\left[-\frac{ik^2}{6m\hbar}(\Delta t)^3-\frac{ik}{\hbar}\Delta t.z_a\right]\int dp'\exp[-\frac{i}{2m\hbar}\Delta t.p'^2+\frac{}{\hbar}\left(z_b-z_a-\frac{k}{2m}(\Delta t)^2\right)p'],$$

where we can apply (7) to determine the following propagator [1, 4]:

$$K(z_bt_b,z_at_a)=\sqrt{\frac{m}{2\pi\hbar(t_b-t_a)}}\exp\{\frac{i}{2\hbar}\left[\frac{m}{2\hbar}(z_a-z_b)^2-k(z_a+z_b)(tb-t_a)-\frac{k^2}{12m}(t_b-t_a)^3\right]\}. \tag{14}$$

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